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Ghost constraints and the covariant quantization of the superparticle in ten dimensions

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Abstract: We present a modification of the Berkovits superparticle. This is firstly in order to covariantly quantize the pure spinor ghosts, and secondly to covariantly calculate matrix elements of a generic operator between two states. We proceed by lifting the pure spinor ghost constraints and regaining them through a BRST cohomology. We are then able to perform a BRST quantization of the system in the usual way, except for some interesting subtleties. Since the pure spinor constraints are reducible, ghosts for ghosts terms are needed, which have so far been calculated up to level 4. Even without a completion of these terms, we are still able to calculate arbitrary matrix elements of a physical operator between two physical states.

Keywords: BRST Quantization, Superstrings and Heterotic Strings, BRST Symmetry, Superspaces.

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1. Introduction

The Brink-Schwarz superparticle action [1] yields a manifestly super-Poincaré covariant, classical description of a free particle moving in superspace. However, covariant quantization has so far proved problematic.

Recently, Berkovits and collaborators have proposed a separate, super-Poincaré covariant model for the $D = 10$, $N = 1$ superparticle [2, 3] which began initially as a superstring model [4]–[10]. Also, in [11], the heterotic Berkovits string was derived from the $n = 2$ superembedding formulation, and an alternative covariant approach without pure spinors was suggested in [12, 13, 14], which we discuss in more detail in section 3.2.

The approach of Berkovits is derives from work by Howe [15, 16], in which an on-shell superspace description of $D = 10$ super Yang-Mills and supergravity is given as integrability conditions along pure spinor lines. Berkovits found a BRST operator, which we refer to as $\hat{Q}$, with pure spinors as ghosts. The ghost number one state cohomology of $\hat{Q}$ is covariant and corresponds exactly with the quanta of on-shell super-Maxwell theory, which is the correct spectrum for the $D = 10$, $N = 1$ superparticle. While this is pleasing, there are unsolved problems which we attempt to address in this article.

Firstly, there is a difficulty in finding a covariant description of the physical degrees of freedom of the pure spinor ghost and its conjugate momentum. Secondly, we require an inner product on the Hilbert space. Thirdly, a systematic study of the space of physical operators of the theory is needed, and a direct comparison to the physical operators of the light-cone gauge Brink-Schwarz superparticle should be made. Fourthly, there is an issue that the Berkovits BRST operator is not hermitian. Our findings on these problems are now discussed below.

1. Covariance: we argue later in section 3 that the pure spinor constraints should be treated as first class, i.e. as gauge generators. In order to describe the physical degrees of freedom of the constrained ghosts, one approach, as described in appendix D, is to completely gauge fix and solve the combined pure spinor and gauge fixing constraints explicitly using U(5) co-ordinates, after first Wick-rotating from SO(9,1) to SO(10). This approach has much in common with taking the light-cone gauge for the bosonic particle. In both cases, only the physical degrees of freedom remain after gauge fixing.

Also, Lorentz invariance is broken, in the case of pure spinors from SO(10) to U(5), and in the case of the particle to SO(8).

It seems natural, just as with the bosonic particle, to attempt a BRST approach in which instead of removing unphysical degrees of freedom, the phase space is expanded with extra ghosts thus maintaining covariance. Physical operators and states are then regained through a BRST cohomology.

There are however extra subtleties involved in imposing ghost constraints as opposed to ordinary constraints. In particular, the ghost constraints cannot be combined into the Berkovits BRST operator. We find that the solution is to introduce a second BRST operator, $\hat{Q}_{gc}$, which simply implements the pure spinor constraints. The Berkovits BRST operator, $\hat{Q}$, then becomes nilpotent modulo $\hat{Q}_{gc}$. The pair of BRST operators form what is known as a BRST double complex.

The pure spinor constraints are reducible, thus ghost for ghost terms are required. So far these terms have been calculated up to level 4. As the number of ghosts for ghosts increases level by level, it seems likely that infinitely many terms will be required. This is not definite though as no pattern has been found, and there is no unique choice of term at each level. Despite this problem, we find that by choosing a suitable representative from each cohomology class, the ghost part factors out in any matrix element calculation of a physical operator between two states. Also, the state cohomology is not affected by this difficulty.

2. The inner product: using the natural Schrödinger measure, the norm of the Berkovits wavefunction is zero, so the question arises of how to define the inner product. Actually, this situation is normal for BRST quantization with no minimal sector using the Schrödinger representation. Take the bosonic particle for example. States in the Berkovits cohomology couple in the inner product to states in a dual cohomology at opposite ghost number. The solution is thus to find a map between the two cohomologies and place states of one cohomology on the left and states of the other on the right within the inner product. In the case of the bosonic particle, in order to obtain the dual ghost number 1/2 wavefunction, you multiply the corresponding ghost number $-1/2$ wavefunction by the c ghost. In general, however, there is no simple map like this, and the dual cohomology has to be explicitly calculated.

3. The operator cohomology: the Berkovits operator cohomology does not correspond with the physical operators of the Brink-Schwarz superparticle. However, we discover that $\hat{Q}$ indirectly implies 'effective constraints', which are not obviously present in the Berkovits BRST operator. These are simply related to the on-shell equations for super Yang-Mills. The operator cohomology modulo these effective constraints matches the physical operators of the Brink-Schwarz superparticle. Thus, the Berkovits and Brink-Schwarz superparticles are equivalent. We also find that these effective constraints are the first class constraints of the Brink-Schwarz particle.

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4. Non-hermicity of $\hat{Q}$: naturally, this can only become an issue once we have defined an inner product. The solution is found in the definition of the inner product, or rather of the dual cohomology.

It should be noted that while issues 2), 3) and 4) are all solved using 1), in principle they can also be studied using U(5) co-ordinates as in appendix D.

The paper is structured as follows. In section 2, we briefly review the Berkovits superparticle model, in section 3 we introduce the idea of the purity constraints being first class, in section 4 the general BRST formulation for theories with first class ghost constraints is detailed. In section 5, a simple example with linear ghost constraints is given, in section 6, we show how $\hat{Q}_{gc}$ is constructed to 4th level. In section 7, we finally build the superparticle model. We also make an analogy with Chern-Simons theory, and compare with the light-cone gauge Brink-Schwarz superparticle. In section 8, we show how our covariant method leads to anomaly cancellation for the open superstring and in section 9 we discuss plans for future research. The appendices mostly consist of relevant reference material. However, note that appendix D on the description of pure spinors using U(5) co-ordinates, is different to the usual Berkovits approach.

2. The $D = 10$, $N = 1$ Berkovits superparticle

The Berkovits, BRST invariant superparticle action [3], which is in hamiltonian form, is given by

$$S_B = \int d\tau \left(\dot{X}^m P_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha w_\alpha - \frac{1}{2} P_m P^m \right), \tag{2.1}$$

where variables X^m, θ^α are the usual $D = 10$, $N = 1$ superspace co-ordinates, and P_m, p_α their conjugate momenta with $m = 1, \ldots, 10$ and $\alpha = 1, \ldots, 16$. Also, θ^α and p_α are fermionic, Majorana-Weyl spinors of opposite chirality. The ghosts λ^α and w_α are bosonic, complex, Weyl spinors with ghost numbers 1 and -1 respectively. The notation used for $D = 10$ spinors and gamma matrices is described in appendix C.

Berkovits defines a BRST operator

$$\hat{Q} = \hat{\lambda}^\alpha \hat{d}_\alpha \,, \qquad \hat{Q}^2 = \hat{P}_m \hat{\lambda}^\alpha \gamma^m_{\alpha\beta} \hat{\lambda}^\beta \,, \tag{2.2}$$

where $\hat{d}_\alpha = \hat{p}_\alpha - i\hat{P}_m(\gamma^m \hat{\theta})_\alpha$ are the fermionic constraint functions of the Brink-Schwarz superparticle, and where $\hat{\lambda}^\alpha$ are defined to obey 'purity' constraints

$$\hat{\lambda}^\alpha \gamma^m_{\alpha\beta} \hat{\lambda}^\beta = 0 \,, \tag{2.3}$$

in order that $\hat{Q}$ be nilpotent. As shown in appendix D, the purity constraints (2.3) leave λ^α with 11, complex degrees of freedom.

Given ghost number operator $\hat{G}' = i\hat{\lambda}^\alpha \hat{w}_\alpha$, the ghost number one, state cohomology $H^1_{st}(\hat{Q})$ describes the physical modes of super-Maxwell theory in a superspace covariant way. Using the Schrödinger representation, we find

$$Q\psi = 0 \,, \qquad \delta\psi = Q\phi(X, \theta) \tag{2.4}$$

$$\Rightarrow \qquad \gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0 \,, \qquad \delta A_\alpha = D_\alpha \phi \tag{2.5}$$

where $\psi = \lambda^\alpha A_\alpha(X,\theta)$ and $\phi(X,\theta)$ are generic ghost number one, and ghost number zero wavefunctions respectively, and where D_α is the usual covariant superspace derivative

$$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} - i\gamma^m_{\alpha\beta}\theta^\beta\partial_m, \;\; \hat{d}_\alpha \equiv -iD_\alpha\,. \tag{2.6}$$

We have also used the identity

$$\lambda^\alpha\lambda^\beta = \frac{1}{16}\lambda\gamma^m\lambda\gamma_m^{\alpha\beta} + \frac{1}{16\cdot 5!}\lambda\gamma^{mnpqr}\lambda\gamma_{mnpqr}^{\alpha\beta}\,, \tag{2.7}$$

which comes from equation (C.10), recalling that $\gamma_{mnp}^{\alpha\beta}$ is antisymmetric in α and β, and so does not contribute. Equation (2.5) describes the equations of motion and gauge transformation for $D=10$, $N=1$ super-Maxwell theory. Therefore $H^1_{st}(\hat{Q})$ corresponds exactly with the spectrum of the Brink-Schwarz superparticle in the light-cone gauge.

3. Pure spinor ghost constraints as first class Dirac constraints

3.1 The BRST double complex

In order to obtain the equations of motion for Berkovits' superparticle, we should solve $\delta S_B = 0$ on the constraint surface $\lambda\gamma^m\lambda$=0. Equivalently, we define an action

$$S = \int d\tau\left(\dot{X}^m P_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha w_\alpha - \frac{1}{2}P_m P^m - \Lambda_m\lambda\gamma^m\lambda\right), \tag{3.1}$$

where Λ_m are ghost number -2, Lagrange multipliers, and then solve $\delta S = 0$ globally. Since $[\lambda\gamma^m\lambda, \lambda\gamma^n\lambda] = 0$, S has the gauge symmetries

$$\delta_\varepsilon\lambda^\alpha = 0\,, \qquad \delta_\varepsilon w_\alpha = -2\varepsilon_m(\gamma^m\lambda)_\alpha\,, \qquad \delta_\varepsilon\Lambda_m = \dot{\varepsilon}_m\,, \tag{3.2}$$

where $\varepsilon_m(\tau)$ is a local, bosonic, ghost number -2 parameter. Thus, the purity constraints can be interpreted as first class constraints. Observables should be gauge invariant with respect to the ghost constraints, as well as $\hat{Q}$-closed, as already argued by Berkovits [3].

In order to covariantly quantize the ghosts, a BRST implementation of the gauge generators $\lambda\gamma^m\lambda$ is required. We define a separate BRST operator $\hat{Q}_{gc}$, with its own associated anti-hermitian ghost number operator $\hat{G}_{gc}$

$$\hat{Q}_{gc} = \hat{C}_m\hat{\lambda}\gamma^m\hat{\lambda} + \cdots\,, \qquad \hat{G}_{gc} = \frac{i}{2}(\hat{C}_m\hat{B}^m - \hat{B}^m\hat{C}_m + \cdots), \tag{3.3}$$

where $\hat{C}_m$ and $\hat{B}^m$ are a fermionic conjugate pair of ghosts. Also, the ellipses refer to ghost for ghost terms, which are needed because the pure spinor constraints are reducible. Calculation of these terms is discussed in section 6. It should be noticed that the Berkovits ghosts λ^α and w_α commute with $\hat{G}_{gc}$, and hence from from the point of view of $\hat{Q}_{gc}$ are treated as ordinary ghost number zero variables. Using this approach, λ^α and w_α are unconstrained and the pure spinor constraints are realized through requiring physical operators and states to belong to the operator or state cohomology $H(\hat{Q}_{gc})$.

Since λ^α are unconstrained, $\hat{Q}$ is no longer nilpotent. However, we say that $\hat{Q}$ is a BRST operator modulo $\hat{Q}_{gc}$, since

$$[\hat{Q}, \hat{Q}_{gc}] = 0\,, \qquad \hat{Q}^2 = [-i\hat{B}^m\hat{P}_m, \hat{Q}_{gc}]\,. \tag{3.4}$$

The first equation of (3.4) implies that $\hat{Q}$ maps any cohomology class of $H(\hat{Q}_{gc})$ onto another one, and the second that $\hat{Q}$ is nilpotent within the phase-space defined by the cohomology $H(\hat{Q}_{gc})$. Physical operators and states are given by the cohomology of $\hat{Q}$ within the cohomology of $\hat{Q}_{gc}$, which is denoted as $H(\hat{Q}|H(\hat{Q}_{gc}))$.

A pair of operators obeying the same general algebra as $\hat{Q}$ and $\hat{Q}_{gc}$ is known as a double complex. This construction is common in mathematics, for example in the calculation of equivariant integrals. However, except for a mention in [17], this type of double complex does not seem to have been explored before in the context of two BRST operators.

In the remainder of this article, we describe how to implement this BRST double complex for a generic constrained ghost system, and in particular for the Berkovits superparticle.

3.2 Discussion of alternative approaches

As a non-covariant alternative to quantizing the pure spinor ghosts, we show in appendix D[1] how to construct canonical, gauge-fixing constraints [18], using U(5) co-ordinates, which completely fix the gauge symmetry generated by $\lambda\gamma^m\lambda$ in a certain region $\lambda^+ \neq 0$, where λ^+ is defined in the appendix. This is similar in approach to imposing the light-cone gauge for the bosonic particle for example. There is an obvious disadvantage here that the Lorentz covariance is reduced to U(5) covariance and also that it's not valid for $\lambda^+ = 0$.

One might expect to be able to build a single covariant BRST operator Q with single ghost number, which implements both the Berkovits BRST operator (2.2) and the pure spinor constraints

$$Q = \lambda^\alpha d_\alpha + b_m\lambda\gamma^m\lambda + \cdots\,, \tag{3.5}$$

where b_m are fermionic ghosts with ghost number -1 and where the dots refer to 'closure' terms, whose sole purpose is to ensure nilpotence of Q off-shell, as opposed to 'physical' terms, which specify the gauge generators. However, it is impossible for Q to be nilpotent without $b_m\lambda\gamma^m\lambda$ becoming a 'closure' term and without introducing new, 'physical', gauge generator terms, which change the physical nature of the theory. It is then wrong to think of this term as introducing ghost constraints, since constraints are implemented only in the 'physical' terms.

Having said this, in the approach taken by Van Nieuwenhuizen and collaborators [12, 13, 14] for the covariant quantization of the superstring, they essentially begin with the above BRST operator (3.5) and through some procedure introduce new 'physical' and 'closure' terms until Q becomes nilpotent. Now $b_m\lambda\gamma^m\lambda$ becomes a 'closure' term, thus there are no longer pure spinor constraints. The resultant BRST operator isn't directly

[1]Our approach to U(5) co-ordinates differs from that of Berkovits [4, 9], which isn't equivalent to a canonical gauge fixing of the first class pure spinor constraints, at least not with initial Poisson bracket $[\lambda^\alpha, w_\beta] = \delta^\alpha_\beta$. Our approach has the advantage of not requiring the fermionization of ghosts.

equivalent to the Berkovits BRST operator due to the extra gauge generators, in fact its cohomology is null. Remarkably though, by restricting physical Vertex operators further to a certain subspace of all possible operators, the cohomology on this subspace, known as an equivariant cohomology, has been shown to be equivalent to the Berkovits cohomology for the open superstring, at least in the massless sector. The advantage of this approach is that there are no pure spinors, and hence this bypasses the problem of a covariant description of them. There is however an issue which needs clarification, which is why there is no central charge cancellation in ten dimensions.

4. Formal description of the method for arbitrary, first class ghost constraints

We now describe the general formulation for the operator quantization of a BRST system with first class ghost constraints. A path integral formulation is also provided in [19]. No attempt will be made at this stage to interpret the constrained ghost quantum system, nor to determine suitable ghost constraints in the general case, since this depends on the particular system in question. Rather, we assume that the defining operators, which are the BRST invariant, bosonic hamiltonian $\hat{H}$ and the two, fermionic BRST operators $\hat{Q}$ and $\hat{Q}_{gc}$, have already been constructed and we proceed to build the generic quantum system from them. Given ghost constraint, BRST operator $\hat{Q}_{gc}$ and corresponding ghost number operator $\hat{G}_{gc}$

$$\hat{Q}_{gc}^2 = 0\,, \qquad [\hat{G}_{gc}, \hat{Q}_{gc}] = \hat{Q}_{gc}\,, \tag{4.1}$$

the operators $\hat{Q}$ and $\hat{G}$ are $\hat{Q}_{gc}$-closed, since they must map between cohomology classes of $H(\hat{Q}_{gc})$

$$[\hat{Q}, \hat{Q}_{gc}] = 0\,, \qquad [\hat{G}, \hat{Q}_{gc}] = 0\,. \tag{4.2}$$

Also

$$\hat{Q}^2 \simeq 0\,, \qquad [\hat{G}, \hat{Q}] \simeq \hat{Q}\,, \tag{4.3}$$

where

$$\hat{A} \simeq \hat{B} \;\Rightarrow\; \hat{A} = \hat{B} + [\hat{C}, \hat{Q}_{gc}]\,, \tag{4.4}$$

for some operator $\hat{C}$. We will only consider theories where $\hat{G}$ and $\hat{G}_{gc}$ commute, so that states can have both ghost numbers well-defined,

$$[\hat{G}, \hat{G}_{gc}] = 0\,. \tag{4.5}$$

Physical operators belong to the ghost number zero, operator cohomology of $\hat{Q}$ within the ghost constraint ghost number zero, operator cohomology of $\hat{Q}_{gc}$, which we denote $H^0_{op}(\hat{Q}|H^0_{op}(\hat{Q}_{gc}))$. A physical operator $\hat{V}$ thus obeys

$$[\hat{V}, \hat{Q}_{gc}] = 0\,, \qquad [\hat{V}, \hat{Q}] \simeq 0 \tag{4.6}$$

$$[\hat{V}, \hat{G}_{gc}] = 0\,, \qquad [\hat{V}, \hat{G}] \simeq 0\,. \tag{4.7}$$

There are two types of BRST-exact term that one can add to $\hat{V}$ in a BRST transformation

$$\hat{V} \sim \hat{V} + [\hat{U}, \hat{Q}] + [\hat{U}_{gc}, \hat{Q}_{gc}] \qquad \text{given } [\hat{U}, \hat{Q}_{gc}] = 0 \,. \tag{4.8}$$

Similarly, a physical state ψ belongs to the ghost number g, state cohomology of $\hat{Q}$ within the ghost constraint ghost number k, state cohomology of $\hat{Q}_{gc}$, which we denote $H^g_{st}(\hat{Q}|H^k_{st}(\hat{Q}_{gc}))$. Thus, ψ satisfies

$$\hat{Q}_{gc}\psi = 0\,, \qquad \hat{Q}\psi \simeq 0 \tag{4.9}$$

$$\hat{G}_{gc}\psi = k\psi\,, \qquad \hat{G}\psi \simeq g\psi\,, \tag{4.10}$$

where the ghost numbers g and k depend on various factors, for instance whether a Schrödinger or Fock representation is being used. Again, there are two types of BRST-exact terms that one can add to ψ in a BRST transformation

$$\psi \sim \psi + \hat{Q}\chi + \hat{Q}_{gc}\chi_{gc} \qquad \text{given } \hat{Q}_{gc}\chi = 0 \,. \tag{4.11}$$

The BRST invariant hamiltonian $\hat{H}$ is a physical, hermitian operator which belongs to $H^0_{op}(\hat{Q}|H^0_{op}(\hat{Q}_{gc}))$, and thus obeys equations (4.6) and (4.7). It is uniquely defined up to BRST-exact terms

$$\hat{H} \sim \hat{H} + [\hat{\chi}_{gc}, \hat{Q}_{gc}] + [\hat{\chi}, \hat{Q}]\,, \qquad \text{where } [\hat{\chi}, \hat{Q}_{gc}] = 0\,, \tag{4.12}$$

where $\hat{\chi}_{gc}$ and $\hat{\chi}$ are gauge-fixing fermions.

In general we require

$$\hat{Q}_{gc} = \hat{Q}^{\dagger}_{gc} \qquad \hat{Q} = \hat{Q}^{\dagger}\,, \tag{4.13}$$

in order that if $\hat{Q}_{gc}\psi = 0$, then $(\psi, \hat{Q}_{gc}\phi) = 0$ for arbitrary state ϕ, where $(\cdot,\cdot)$ is the inner product on the Hilbert space, and where a similar result applies for $\hat{Q}$. However, in the special case of the Berkovits superparticle $\hat{Q} \neq \hat{Q}^{\dagger}$ and $\hat{Q}_{gc} \neq \hat{Q}^{\dagger}_{gc}$, since $\hat{\lambda}^{\alpha} \neq \hat{\lambda}^{\alpha\dagger}$ because λ^{α} are classically complex. This problem is solved in section 7.3.2, in short by taking the complex conjugate of a wavefunction before placing it to the left in the inner product.

The most important consequence of (4.13) is that BRST exact operators have vanishing matrix elements between physical states

$$(\psi_m, [\hat{U}_{gc}, \hat{Q}_{gc}]\psi_n) = 0\,, \qquad \text{for } \psi_m \in H^k_{st}(\hat{Q}_{gc}) \tag{4.14}$$

$$(\psi_m, [\hat{U}, \hat{Q}]\psi_n) = 0\,, \qquad \text{for } \psi_m \in H^g_{st}(\hat{Q}|H^k_{st}(\hat{Q}_{gc}))\,, \tag{4.15}$$

where we have also used that $[\hat{U}, \hat{Q}_{gc}] = 0$ and equation (4.9).

As a final observation, any operator $\hat{A}$ belonging to $H_{op}(\hat{Q}_{gc})$ can be meaningfully expressed as the matrix $(\psi_m, \hat{A}\psi_n)$, where $\{\psi_m\}$ form a basis for $H_{st}(\hat{Q}_{gc})$. When equations involving such operators are written in matrix form, then the symbol $\simeq$ can be replaced with an equals sign. For example,

$$(\psi_m, \hat{Q}^2\psi_n) = 0\,, \qquad \text{given } \psi_m \in H_{st}(\hat{Q}_{gc})\,. \tag{4.16}$$

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5. An example of linear ghost constraints

We now illustrate the formal description of the last section with a simple example. We show how a BRST system with linear ghost constraints, specified by $\hat{Q}_{gc}$, can be related to a gauge theory with a single BRST operator.

A simple example. Consider the motion of a particle described by the action

$$S = \int_{t_1}^{t_2} dt \left(-\frac{1}{2}(q^1)^2 + \frac{1}{2}(\dot{q}^2)^2 - \frac{1}{2}(q^2)^2 \right) . \tag{5.1}$$

The Dirac-Bergmann algorithm yields second class constraints

$$p_1 = 0 , \qquad q^1 = 0 , \tag{5.2}$$

where the first is primary and the second is secondary. The canonical hamiltonian is

$$H = \frac{1}{2}(q^1)^2 + \frac{1}{2}(p_2)^2 + \frac{1}{2}(q^2)^2 . \tag{5.3}$$

By using the Dirac bracket, or simply by parameterizing the constraint surface using only co-ordinates (q^2, p_2), quantization is straightforward.

However, if we were to naively treat the constraints as if they were first class, we could construct the BRST operator

$$\hat{Q} = \hat{\eta}^1 \hat{q}^1 + \hat{\eta}^2 \hat{p}_1 , \qquad \hat{Q}^2 = i\hat{\eta}^1 \hat{\eta}^2 , \tag{5.4}$$

where $(\hat{\eta}^1, \hat{\mathcal{P}}_1)$ and $(\hat{\eta}^2, \hat{\mathcal{P}}_2)$ are fermionic ghost, ghost momenta pairs. Supposing the first class ghost constraint $\hat{\eta}^1 = 0$ is introduced for example, by defining nilpotent BRST operator $\hat{Q}_{gc}$, and its corresponding anti-hermitian ghost operator $\hat{G}_{gc}$

$$\hat{Q}_{gc} = \hat{U}\hat{\eta}^1 , \qquad \hat{G}_{gc} = \frac{i}{2}(\hat{U}\hat{V} + \hat{V}\hat{U}) \tag{5.5}$$

where $\hat{U}$ and $\hat{V}$ define a bosonic, ghost conjugate pair. The anti-hermitian, ghost number operator $i/2(\hat{\eta}^1\hat{\mathcal{P}}_1 - \hat{\mathcal{P}}_1\hat{\eta}^1 + \hat{\eta}^2\hat{\mathcal{P}}_2 - \hat{\mathcal{P}}_2\hat{\eta}^2)$ is gauge invariant with respect to constraint $\hat{\eta}^1 = 0$, but requires a BRST extension in order to become $\hat{Q}_{gc}$-closed

$$\hat{G} = \frac{i}{2}\left(\hat{\eta}^1\hat{\mathcal{P}}_1 - \hat{\mathcal{P}}_1\hat{\eta}^1 + \hat{\eta}^2\hat{\mathcal{P}}_2 - \hat{\mathcal{P}}_2\hat{\eta}^2 - (\hat{U}\hat{V} + \hat{V}\hat{U}) \right) . \tag{5.6}$$

We can now verify that the above definitions of $\hat{Q}_{gc}$, $\hat{Q}$, $\hat{G}_{gc}$ and $\hat{G}$ obey the required equations (4.1) to (4.5).

Now let us calculate the classical, physical functions. Since $Q \simeq \eta^2 p_1$, we deduce that

$$H^0(Q|H^0(Q_{gc})) \cong H^0(\eta^2 p_1 | C^\infty(q^i, p_i, \eta^2, \mathcal{P}_2)) \cong C^\infty(q^2, p_2) . \tag{5.7}$$

In effect, the ghosts U and V have cancelled the ghosts η^1 and $\mathcal{P}_1$, thus establishing equivalence to an ordinary BRST theory, with BRST operator $\eta^2 p_1$ in phase-space defined by canonical co-ordinates $(q^i, p_i, \eta^2, \mathcal{P}_2)$.

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In this particular example, introducing the ghost constraint $\eta^1 = 0$ is equivalent to removing the second class constraint $q^1 = 0$, which one can think of as a canonical gauge-fixing constraint, thus leaving only first class constraint $p_1 = 0$. The process of removing canonical gauge-fixing constraints is sometimes known as 'gauge unfixing' [20, 21, 22], which is also similar to 'split involution' [17]. There is some similarity between our approach here and the recent projection operator approach to the BRST quantization of general constrained systems by Batalin et al. [23]. There they also introduce ghosts for all the second class constraints, for the purpose of covariance. The extra ghost degrees of freedom are then cancelled by adding ghost for ghost terms to the BRST operator.

The hamiltonian isn't BRST invariant, since it isn't gauge invariant with respect to p_1 because $[H, p_1] = -q^1$ isn't zero on $p_1 = 0$. Thus, we replace H with gauge invariant hamiltonian $\tilde{H}$, which possesses the properties

$$\tilde{H}|_{q^1=p_1=0} = H|_{q^1=p_1=0}\,, \qquad [\tilde{H}, p_1]|_{p_1=0} = 0\,. \tag{5.8}$$

A suitable choice is

$$\tilde{H} = \frac{1}{2}(q^2)^2 + \frac{1}{2}(p_2)^2\,, \tag{5.9}$$

which is Q_{gc}-closed without need for further BRST extension.

We now calculate states and operators in the Schrödinger representation in order to observe how the classical equivalence shown in equation (5.7), is extended to a quantum mechanical equivalence. It is simpler to notice first that

$$\hat{G} \simeq \frac{i}{2}(\hat{\eta}^2\hat{\mathcal{P}}_2 - \hat{\mathcal{P}}_2\hat{\eta}^2)\,, \qquad \hat{Q} \simeq \hat{\eta}^2\hat{p}_1\,. \tag{5.10}$$

The 'physical' states in the ghost constraint cohomology appear at ghost constraint ghost numbers $-1/2$ and $+1/2$ defined by states $\psi_{U=0}$ and $\psi_{V=0}$ respectively, where in this notation,

$$\hat{U}\psi_{U=0} = 0\,, \qquad \hat{V}\psi_{V=0} = 0\,, \qquad (\psi_{U=0}, \psi_{V=0}) = 1\,. \tag{5.11}$$

Thus, a basis for $\hat{Q}_{gc}$-closed wavefunctions with ghost constraint ghost numbers $\pm 1/2$ is given by

$$\psi_{-1/2}(U, \eta^1) = \delta(U)(a\eta^1 + b)\,, \qquad \psi_{1/2}(U, \eta^1) = c\eta^1\,, \tag{5.12}$$

where a, b and c are c-number constants. Note that a can be transformed to zero by adding $\hat{Q}_{gc}$-exact state $\hat{Q}_{gc}\delta'(U)a$ to $\psi_{1/2}$. A generic, physical wavefunction $\psi \in H_{st}^{\pm\frac{1}{2}}(\hat{Q}|H^{\pm\frac{1}{2}}(\hat{Q}_{gc}))$ can be written as

$$\psi = \psi_{\pm\frac{1}{2}}(U, \eta^1)\psi_m(\eta^2, q^1, q^2)\,, \qquad \text{given } \hat{Q}\psi_m(\eta^2, q^1, q^2) = 0\,. \tag{5.13}$$

All operators $\hat{F} \in H^0_{op}(\hat{Q}|H^0_{op}(\hat{Q}_{gc}))$ can be written in the form

$$\hat{F} = \hat{F}_1(\hat{\eta}^2, \hat{\mathcal{P}}_2, \hat{q}^i, \hat{p}_i) + \hat{F}_2(\hat{U}, \hat{V}, \hat{\eta}^i, \hat{\mathcal{P}}_i, \hat{q}^i, \hat{p}_i) \simeq \hat{F}_1(\hat{\eta}^2, \hat{\mathcal{P}}_2, \hat{q}^i, \hat{p}_i)\,. \tag{5.14}$$

This is because $\hat{F}_1$ is separately $\hat{Q}_{gc}$-closed, being independent of $\hat{V}$ and $\hat{\mathcal{P}}_1$, and since $H^0(Q_{gc}) \cong C^\infty(q^i, p_i, \eta^2, \mathcal{P}_2)$, $\hat{F}_2$ must be $\hat{Q}_{gc}$-exact. Therefore the most general matrix element of physical operator $\hat{F}$ between two physical states is

$$(\psi_{\frac{1}{2}}(U,\eta^1)\psi_m(\eta^2,q^1,q^2),\, \hat{F}\psi_n(\eta^2,q^1,q^2)\psi_{-\frac{1}{2}}(U,\eta^1)) = \\ = (\psi_m(\eta^2,q^1,q^2),\, \hat{F}_1(\hat{\eta}^2,\hat{\mathcal{P}}_2,\hat{q}^i,\hat{p}_i)\psi_n(\eta^2,q^1,q^2)) \,, \quad (5.15)$$

given $b = c = 1$, where to obtain the second line, we have integrated out η^1 and U in the Schrödinger inner product. Thus, we have seen how quantum mechanically, the above system is equivalent to a gauge theory with single BRST operator $\hat{Q} = \hat{\eta}^2\hat{p}_1$ and phase space co-ordinates $(\hat{q}^i, \hat{p}_i, \hat{\eta}^2, \hat{\mathcal{P}}_2)$.

6. The pure spinor BRST operator $\hat{Q}_{gc}$

As already mentioned in section 3, the pure spinor constraints $\lambda\gamma^m\lambda = 0$ are reducible and hence $\hat{Q}_{gc}$ requires ghost for ghost terms. Unfortunately, only terms up to the fourth level of ghosts for ghosts have been found so far. It is not yet known whether a covariant termination will exist or whether infinite ghosts for ghosts will be needed. In section 7, we nominally specify $\hat{Q}_{gc}$ up to the 2nd level of ghosts, which is just high enough in order to spot any patterns in the BRST extension of $\hat{Q}_{gc}$-closed operators. Without a full solution, there is no advantage in specifying $\hat{Q}_{gc}$ to the highest known level of ghosts.

We proceed with a brief recipe describing how to build a generic BRST operator with reducible constraints [18], and then build the reducibility identities up to level 4 and $\hat{Q}_{gc}$ up to level 2.

6.1 A recipe for the construction of a BRST charge with reducible constraints

We begin with a set of first class constraints

$$g_{a_0} = 0 \qquad a_0 = 1, \dots, m_0 \,, \quad (6.1)$$

of which $m \le m_0$ are independent. We define Z_1 with the properties

$$(Z_1)_{a_1}{}^{a_0} g_{a_0} = 0 \,, \qquad a_1 = 1, \dots, m_1 \quad (6.2)$$

$$\text{Rank}\,(Z_1)_{a_1}{}^{a_0} \approx m_0 - m \,, \quad (6.3)$$

where $A \approx B$ implies that A is equal to B on the constraint surface $g_{a_0} = 0$. So Z_1 not only annihilates g_{a_0} globally in phase-space, but also describes all $(m_0 - m)$ vanishing linear combinations of the gauge generators locally on the constraint surface, since $(Z_1)_{a_1}{}^{a_0}[g_{a_0}, F] \approx 0$, for any function F. The order k reducibility identity describes all vanishing linear combinations of Z_{k-1}

$$(Z_k)_{a_k}{}^{a_{k-1}}(Z_{k-1})_{a_{k-1}}{}^{a_{k-2}} \approx 0 \,, \qquad a_k = 1, \dots m_k \,. \quad (6.4)$$

where

$$\text{Rank}\,(Z_k)_{a_k}{}^{a_{k-1}} \approx m'_k \,, \qquad m'_k = m_{k-1} - m'_{k-1} \,. \quad (6.5)$$

We keep building more Z_k's until there are no vanishing combinations left, i.e. until Rank $Z_k \approx m_k$, or until we establish a pattern if there are infinite Z_k's. For theories with finite reducibility level L, we can express the number of independent constraints m as

$$m = \sum_{i=0}^{L} (-)^i m_i \,. \tag{6.6}$$

We introduce conjugate ghost pairs for every Z_k as follows

$$\eta^{a_k}, \varepsilon(\eta^{a_k}) = \varepsilon_{a_0} + k + 1\,, \qquad \mathrm{gh}\,\eta^{a_k} = 1 + k \tag{6.7}$$

$$\mathcal{P}_{a_k}, \varepsilon(\eta^{a_k}) = \varepsilon_{a_0} + k + 1\,, \qquad \mathrm{gh}\,\mathcal{P}_{a_k} = -1 - k\,, \tag{6.8}$$

where ε describes the Grassmann number and 'gh', the ghost number. Also, the Poisson bracket is as usual

$$[\mathcal{P}_{a_k}, \eta^{b_j}] = (-)^{\varepsilon_{a_0}+k} \delta^{b_k}_{a_k} \delta_{kj}\,. \tag{6.9}$$

We finally define the boundary terms in the ghost number one, fermionic BRST charge

$$Q = \eta^{a_0} g_{a_0} + \sum_{k=1}^{L} \left(\eta^{a_k} (Z_k)_{a_k}{}^{a_{k-1}} \mathcal{P}_{a_k - 1}\right) + \text{'more'}\,, \tag{6.10}$$

where it can be shown that the requirement $[Q, Q] = 0$ determines the rest of the terms, and that Q is unique up to a canonical transformation.

6.2 Construction of the pure spinor BRST operator

As shown in appendix D, the 16 component, pure spinor λ^α possesses exactly 11 independent complex degrees of freedom. This means that of the 10, complex, pure spinor constraints, only 5 are independent and ghost for ghost terms are needed in Q_{gc}. Information required to build Q_{gc} to level 4 is summarized in table 1.

We define k, m_k and Z_k as in section 6.1, where Z_0 corresponds to the pure spinor constraints. The rank of the reducibility matrix Z_k is calculated on-shell and Z_k must be chosen such that Rank Z_k is exactly equal to the number of redundant linear combinations contained in Z_{k-1} as in equation 6.5. Ghosts denoted by (C, B) are fermionic conjugate pairs and those by (U, V) are bosonic. We define ε to be the Grassmann parity of the ghosts and gh_{gc} the ghost constraint ghost number.

Level k	Z_k	Rank Z_k	m_k	ghosts	gh_{gc}	ε
0	$\lambda\gamma^m\lambda$	5	10	C_m, B^m	$1, -1$	1
1	$(\gamma_m\lambda)_\alpha$	5	16	U^α, V_α	$2, -2$	0
2	$(\lambda\gamma_n\gamma_p)^\alpha$	11	46	C^{np}, B_{np}	$3, -3$	1
3	$(\lambda\gamma^{[n})_\beta\delta^{p]}_q + \frac{1}{10}(\lambda\gamma_q)_\beta\eta^{np}$	35	160	$U^{\beta q}, V_{\beta q}$	$4, -4$	0
4	$(\lambda\gamma^{rs})^\beta\eta^{tq}$	125	450	C_{rst}, B^{rst}	$5, -5$	1
4	$\delta^\beta_{(\eta}(\lambda\gamma^q)_{\gamma)}$	n/a	136	$C^{\eta\gamma}, B_{\eta\gamma}$	$5, -5$	1

Table 1: Reducibility of pure spinor constraints.

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As there appears to be no obvious analytical way of deriving ranks, they were calculated numerically for particular pure values of λ^α, using the computer software package Maple. It is expected that the ranks remain constant for all pure values of λ^α, though no proof of this is provided here.

There are some further subtleties here. Notice that C^{np} has 46 components, since it consists of an antisymmetric piece, with 45 components, and a trace piece with just one component. Note also that Z_3 consists of the sum of the antisymmetric and trace piece, with respect to n, p, of $(\lambda\gamma^n)_\beta \delta^p_q$.

Below are the reducibility identities, the first of which (6.11) completely describes the redundancy in the constraints as in equation (6.2). Successive identities take the form $Z_k Z_{k-1} \approx 0$ as in (6.4). The complete equations for $Z_4 Z_3 \approx 0$ consist of two identities (6.14) and (6.15), whereas the rest require just one equation. The gamma matrix identities in appendix C may be used to confirm them.

$$(\gamma_m\lambda)_\alpha \lambda\gamma^m\lambda = 0\,, \tag{6.11}$$

$$(\lambda\gamma_n\gamma_p)^\alpha(\lambda\gamma_m)_\alpha = \lambda\gamma_m\lambda\eta_{np} + \lambda\gamma_n\lambda\eta_{pm} - \lambda\gamma_p\lambda\eta_{nm} \approx 0\,, \tag{6.12}$$

$$\left[(\lambda\gamma^{[n})_\beta\delta^{p]}_q + \frac{1}{10}(\lambda\gamma_q)_\beta\eta^{np}\right](\lambda\gamma_n\gamma_p)_\alpha = -\frac{1}{2}\lambda\gamma^n\lambda(\gamma_n\gamma_p)^\alpha_\beta \approx 0\,, \tag{6.13}$$

$$[(\lambda\gamma^{rs})^\beta\eta^{tq}]\left[(\lambda\gamma^{[n})_\beta\delta^{p]}_q + \frac{1}{10}(\lambda\gamma_q)_\beta\eta^{np}\right] \approx 0\,, \tag{6.14}$$

$$[\delta^\beta_{(\eta}(\lambda\gamma^q)_{\gamma)}]\left[(\lambda\gamma^{[n})_\beta\delta^{p]}_q + \frac{1}{10}(\lambda\gamma_q)_\beta\eta^{np}\right] \approx 0\,. \tag{6.15}$$

Let us study the first level reducibility condition (6.11), in detail. We firstly confirm the reducibility condition using the Fierz identity of equation (C.11). There are $m_1 = 16$ linear combinations of $\lambda\gamma^m\lambda$, denoted by $Z_1 = (\gamma_m\lambda)_\alpha$, which disappear globally. We calculate, using Maple, that Rank $(\gamma_m\lambda)_\alpha \approx 5$, so there are only 5 linearly independent combinations, which match the 5 redundant constraints in $\lambda\gamma^m\lambda$. Therefore, $Z_1 = (\gamma_m\lambda)_\alpha$ contains $16 - 5 = 11$ redundant linear combinations of $\lambda\gamma^m\lambda$, which need to be taken care of at the next level.

If $\hat{Q}_{gc}$ were to terminate at finite level L, we could count the number of independent first class ghost constraints by the graded sum in equation (6.6), with $m = 5$. This has an important bearing on the vanishing of the central charge for the superstring, which is mentioned in section 8. In the case of infinite ghosts for ghosts, then the sum would need to be regularized.

The process of finding Z_k's is largely a matter of trial and error. Candidate Z_k's are put forward which annihilate Z_{k-1}, then their ranks are checked using Maple. We restrict the search to Z_k's linear in λ^α, since higher order powers of λ^α tend to have significantly higher m_k, i.e. redundancy, for a given rank.

Finally, we construct the ghost constraint, BRST operator, up to level 2, in the manner of equation (6.10).

$$\begin{aligned}\hat{Q}_{gc} &= \hat{C}_m\hat{\lambda}\gamma^m\hat{\lambda} + \hat{U}^\alpha(\hat{\lambda}\gamma_m)_\alpha\hat{B}^m + \hat{C}^{np}(\hat{\lambda}\gamma_n\gamma_p)^\alpha\hat{V}_\alpha + \cdots \\ &\quad + \left(\hat{B}_n\hat{B}_p\hat{C}^{np} - \frac{1}{2}\hat{B}_m\hat{B}^m\hat{C}^{np}\eta_{np}\right) + \cdots\,,\end{aligned} \tag{6.16}$$

where the expressions in the first line are the boundary terms. The anti-hermitian, ghost constraint ghost number operator is given by

$$\hat{G}_{gc} = \frac{i}{2}\left(\hat{C}_m\hat{B}^m - \hat{B}^m\hat{C}_m + 2\hat{U}^\alpha\hat{V}_\alpha + 2\hat{V}_\alpha\hat{U}^\alpha + 3\hat{C}^{np}\hat{B}_{np} - 3\hat{B}_{np}\hat{C}^{np} + \cdots\right). \qquad (6.17)$$

7. Towards the covariant quantization of the $D = 10$, $N = 1$ superparticle

We proceed to build a quantum system in the Schrödinger representation, in the manner of section 4, for the superparticle theory with Berkovits BRST operator $\hat{Q} = \hat{\lambda}^\alpha\hat{d}_\alpha$, and with first class ghost constraints described by $\hat{Q}_{gc}$ in equation (6.16).

The fact that $\hat{Q}_{gc}$ is incomplete means that we cannot explicitly calculate the BRST extension with respect to $\hat{Q}_{gc}$, of arbitrary operators which are gauge invariant with respect to $\hat{\lambda}\gamma^m\hat{\lambda}$. We tackle this issue in section 7.4.1, by constructing a basis for ghost number zero operators in $H^0_{op}(\hat{Q}_{gc})$, whose properties can be deduced without having to build their respective BRST extensions explicitly. The price to be paid is that we have only one representative of each cohomology class of $H^0_{op}(\hat{Q}_{gc})$.

Our approach is systematic. We build the defining operators in section 7.1, the gauge-fixed action in 7.2, the physical states in 7.3 and the physical operators in 7.4.

In the latter two sections, we begin with the ghost constraint cohomology $H(\hat{Q}_{gc})$ in subsections 7.3.1 and 7.4.1, before the physical cohomology $H(\hat{Q}|H(\hat{Q}_{gc}))$ in 7.3.2 and 7.4.2, and we compare with the Brink-Schwarz model in 7.3.3 and 7.4.3.

We also construct the super-Poincaré covariant, inner product in section 7.3.2 and draw an analogy with the Witten, particle wavefunction for Chern-Simons theory in 7.3.4.

We further discover in subsection 7.4.2 that $\hat{Q}$ indirectly implies what we name as 'effective constraints'. The operator cohomology $H^0_{op}(\hat{Q}|H^0_{op}(\hat{Q}_{gc}))$ modulo these 'effective constraints' seems to correspond with the space of light-cone gauge operators of the Brink-Schwarz model.

There are some useful, relevant results contained in appendix B concerning BRST quantization in the Schrödinger representation.

7.1 The defining operators $\hat{Q}_{gc}$, $\hat{G}_{gc}$, $\hat{Q}$ and $\hat{G}$

As already observed,

$$\hat{Q}^2 = [-i\hat{B}^m\hat{P}_m, \hat{Q}_{gc}] \simeq 0\,, \qquad (7.1)$$

and we have the first terms of the anti-hermitian $\hat{G}_{gc}$ in equation (6.17). However, the ghost number operator $i\hat{\lambda}^\alpha\hat{w}_\alpha$ is gauge invariant

$$[i\hat{\lambda}^\alpha\hat{w}_\alpha, \hat{\lambda}\gamma^m\hat{\lambda}] \approx 0\,, \qquad (7.2)$$

but requires a BRST extension in order to make it Q_{gc}-closed. Working up to reducibility level 2 again, we find

$$\hat{G}' = i\left(\hat{\lambda}^\alpha\hat{w}_\alpha - 2\hat{C}_m\hat{B}^m - 3\hat{U}^\alpha\hat{V}_\alpha - 4\hat{C}^{np}\hat{B}_{np} - \cdots\right), \qquad [\hat{G}', \hat{Q}_{gc}] = 0\,. \qquad (7.3)$$

Similarly, the anti-hermitian ghost number is given by

$$\hat{G} = \frac{i}{2}\Big[\hat{\lambda}^\alpha \hat{w}_\alpha + \hat{w}_\alpha \hat{\lambda}^\alpha - 2(\hat{C}_m \hat{B}^m - \hat{B}^m \hat{C}_m) - 3(\hat{U}^\alpha \hat{V}_\alpha + \hat{V}_\alpha \hat{U}^\alpha) - \\ - 4(\hat{C}^{np}\hat{B}_{np} - \hat{B}_{np}\hat{C}^{np})\cdots\Big]\,. \tag{7.4}$$

Notice also

$$\hat{G} = \frac{i}{2}\left[\hat{\lambda}^\alpha \hat{w}_\alpha + \hat{w}_\alpha \hat{\lambda}^\alpha - (\hat{C}_m \hat{B}^m - \hat{B}^m \hat{C}_m) - (\hat{U}^\alpha \hat{V}_\alpha + \hat{V}_\alpha \hat{U}^\alpha) - \cdots\right] - \hat{G}_{gc}\,. \tag{7.5}$$

Careful calculation reveals that the above relation holds for all levels, at least if all Z_k are linear in λ^α.

We now have $\hat{Q}_{gc}$, $\hat{G}_{gc}$, $\hat{Q}$ and $\hat{G}$, which can be confirmed to obey equations (4.1) to (4.5) as required.

7.2 The gauge-fixed, BRST invariant, superparticle action

Given the action in equation (3.1), it remains to gauge-fix the pure spinor gauge symmetry in order to obtain the full BRST action. If one chooses the gauge $\Lambda_m = 0$, the full action is given by

$$S = \int d\tau \bigg(\dot{X}^m P_m + \dot{\theta}^\alpha p_\alpha - \frac{1}{2}P_m P^m + \dot{\lambda}^\alpha w_\alpha + \\ + \dot{C}_m B^m + \dot{U}^\alpha V_\alpha + \dot{C}^{np} B_{np} + \cdots\bigg)\,. \tag{7.6}$$

It is simplest to think of the gauge-fixing procedure from the hamiltonian point of view, where this corresponds simply to the choice of zero gauge-fixing fermion and hence zero ghost hamiltonian. The intermediate 'first class' hamiltonian, by which one means first class with respect to the ghost constraints, is given by $H = 1/2P_m P^m$, which is already BRST invariant with respect to both Q_{gc} and Q.

7.3 The physical states

7.3.1 The pure spinor, state cohomology $H^{\pm k}_{st}(\hat{Q}_{gc})$

A physical state obeys equations (4.9) and (4.10), belonging to $H_{st}(\hat{Q}|H_{st}(\hat{Q}_{gc}))$. A preliminary step is to obtain wavefunctions in the usual Schrödinger representation, belonging to the state cohomology $H^{\pm k}_{st}(\hat{Q}_{gc})$. The ghost constraint ghost numbers $\pm k$, which are undetermined since $\hat{Q}_{gc}$ hasn't yet been completed, refer to the cohomologies at which the pure spinor constraints are imposed as Dirac constraints.

We specify states $\phi_{C=0,U=0}$ and $\phi_{B=0,V=0}$, which are defined up to a normalization factor by $\hat{C}^m \phi_{C=0,U=0} = \hat{U}^\alpha \phi_{C=0,U=0} = \cdots = 0$ for all $\hat{C}$'s and $\hat{U}$'s and similarly for $\phi_{B=0,V=0}$, where

$$\hat{G}_{gc}\phi_{C=0,U=0} = k\phi_{C=0,U=0}\,, \qquad \hat{G}_{gc}\phi_{B=0,V=0} = -k\phi_{B=0,V=0}\,. \tag{7.7}$$

The wavefunctions are normalized as

$$\phi_{C=0,U=0} = \left(\prod_m C_m\right)\delta^{(16)}(U)\cdots\,, \qquad \phi_{B=0,V=0} = 1\,, \tag{7.8}$$

so that $(\phi_{B=0,V=0}, \phi_{C=0,U=0}) = 1$.

A wavefunction $\psi_-(\lambda)\phi_{B=0,V=0}$ in $H^{-k}_{st}(\hat{Q}_{gc})$ obeys,

$$\hat{Q}_{gc}\psi_-(\lambda)\phi_{B=0,V=0} = 0 \;\Rightarrow\; \hat{\lambda}\gamma^m\hat{\lambda}\psi_-(\lambda) = 0 \tag{7.9}$$

and there are no $\hat{Q}_{gc}$-exact states at this ghost constraint ghost number. On the other hand, a wavefunction $\psi_+(\lambda)\phi_{C=0,U=0}$, in the isomorphic state cohomology $H^k_{st}(\hat{Q}_{gc})$ is $\hat{Q}_{gc}$-closed for any function $\psi_+(\lambda)$, but we can vary the wavefunction by arbitrary $\hat{Q}_{gc}$-exact amounts

$$\delta\psi_+(\lambda)\phi_{C=0,U=0} \;=\; i\hat{Q}_{gc}\, f_m(\lambda)\hat{B}^m\phi_{C=0,U=0} \tag{7.10}$$

$$\Rightarrow\;\; \delta\psi_+(\lambda) \;=\; f_m(\lambda)\lambda\gamma^m\lambda, \tag{7.11}$$

for arbitrary wavefunction $f_m(\lambda)$. As usual, the two cohomologies are isomorphic

$$H^k_{st}(\hat{Q}_{gc}) \cong H^{-k}_{st}(\hat{Q}_{gc})\,. \tag{7.12}$$

Note, the fact that there are ghosts for ghosts doesn't affect the state cohomology. It only becomes important in the operator cohomology, where they are there to cancel the 5 excess degrees of freedom hidden in the 10 ghosts C_m.

See appendix B for a brief discussion of the BRST concepts which have arisen in this subsection.

7.3.2 The physical state cohomologies $H^{\pm(g+1)}_{st}(\hat{Q}|H^{\pm k}_{st}(\hat{Q}_{gc}))$

In the previous subsection, we have not yet worried about requiring $\hat{Q}\psi \simeq 0$ or constraining the generic wavefunctions $\psi_\pm(\lambda)$ to a particular ghost number. We define the ghost part of generic physical wavefunctions as

$$\phi_g = \phi_{w=0,C=0,U=0}\,, \qquad \phi_{-g} = \phi_{\lambda=0,B=0,V=0} \tag{7.13}$$

where,

$$\phi_g \in H^k_{st}(\hat{Q}_{gc})\,, \qquad \phi_{-g} \in H^{-k}_{st}(\hat{Q}_{gc})\,, \tag{7.14}$$

$$\hat{G}\phi_{\pm g} = \pm g\phi_{\pm g}\,, \qquad g = \frac{11}{2} - k\,, \tag{7.15}$$

where the expression for g in terms of k is deduced from (7.5) and (6.6). The wavefunctions are normalized as in equation (7.8), thus $(\phi_{-g}, \phi_g) = 1$. There are two ingredients to each of the states ϕ_g and ϕ_{-g}. In ϕ_{-g} for example, there is firstly a delta-function, which fixes 5 components of λ^α in terms of the other 11, so that equation (7.9) is obeyed. Secondly, there is a delta function to set the remaining 11 components of λ^α to zero, in order to provide the standard ghost number g state. The combined wavefunction $\phi_{\lambda=0}$ is straightforward, however it seems difficult to split it into the two aforementioned parts. The generic ghost number $(g+1)$ wavefunction is given by

$$\psi_{g+1} = \hat{\lambda}^\alpha A_\alpha(X,\theta)\phi_g\,, \tag{7.16}$$

which is simply our version of the Berkovits ghost number one wavefunction. The conditions of BRST invariance (4.9) imply, in a similar manner to Berkovits in section 2, that $A_\alpha(X,\theta)$

obeys the super-Maxwell equations of motion

$$\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0\,, \tag{7.17}$$

since $\hat\lambda^\alpha \hat\lambda^\beta D_\alpha A_\beta(X,\theta)\phi_g \simeq 0$, and $\gamma_m^{\alpha\beta}\hat\lambda\gamma^m\hat\lambda D_\alpha A_\beta \phi_g$ is $\hat Q_{gc}$-exact. Also, ψ_{g+1} is $\hat Q_{gc}$-closed for arbitrary $A_\alpha(X,\theta)$.

The BRST transformation of the wavefunction ψ_{g+1} is

$$\delta\psi_{g+1} = \hat Q\,\Lambda(X,\theta)\phi_g\,, \tag{7.18}$$

where $\Lambda(X,\theta)\phi_g$ is $\hat Q_{gc}$-closed for arbitrary $\Lambda(X,\theta)$, which implies the usual super-Maxwell gauge transformation

$$\delta A_\alpha(X,\theta) = D_\alpha \Lambda(X,\theta)\,. \tag{7.19}$$

Our wavefunction ψ_{g+1} couples in the inner product to certain states at opposite ghost numbers, as in equation (B.1), given by

$$\psi_{(-g-1)} = \hat w_\alpha \tilde A^\alpha(X,\theta)\phi_{-g}\,, \tag{7.20}$$

The conditions of BRST invariance equation (4.9) imply the equation of motion

$$D_\alpha \tilde A^\alpha(X,\theta) = 0\,. \tag{7.21}$$

The BRST transformation of the wavefunction $\psi_{(-g-1)}$ is

$$\delta\psi_{(-g-1)} = \hat Q B^{\alpha\beta}(X,\theta)\hat w_\alpha \hat w_\beta \phi_{-g} \qquad \text{for } \hat Q_{gc} B^{\alpha\beta}(X,\theta)\hat w_\alpha \hat w_\beta \phi_{-g} = 0\,, \tag{7.22}$$

which implies the following gauge transformation

$$\delta\tilde A^\alpha(X,\theta) = D_\beta B^{\alpha\beta}(X,\theta) \qquad \text{for } \gamma^m_{\alpha\beta} B^{\alpha\beta} = 0\,. \tag{7.23}$$

We expect the two cohomologies to be isomorphic

$$H^{(g+1)}_{st}(\hat Q|H^k_{st}(\hat Q_{gc})) \cong H^{(-g-1)}_{st}(\hat Q|H^{-k}_{st}(\hat Q_{gc}))\,, \tag{7.24}$$

and we relate them in section 7.3.3 through the Schrödinger inner product, which we now define.

Since λ^α and w_α are complex, we define

$$\bar\lambda^\alpha = (\lambda^\alpha)^*\,, \qquad \bar w_\alpha = (w_\alpha)^*\,, \tag{7.25}$$

thus,

$$(\hat\lambda^\alpha)^\dagger = \hat{\bar\lambda}^\alpha\,, \qquad (\hat w_\alpha)^\dagger = \hat{\bar w}_\alpha\,. \tag{7.26}$$

Given a state ψ, we define $\bar\psi = \psi^*$. Thus, in particular

$$\bar\psi_{g+1} = \hat{\bar\lambda}^\alpha A_\alpha(X,\theta)\phi_g\,, \qquad \bar\psi_{(-g-1)} = \hat{\bar w}_\alpha \tilde A^\alpha(X,\theta)\phi_{-g}\,, \tag{7.27}$$

where we choose the constant phase factors present in ψ_{g+1} and $\psi_{(-g-1)}$ such that $A_\alpha(X,\theta)\phi_g$ and $\tilde{A}^\alpha(X,\theta)\phi_{-g}$ are both real. We find that by replacing $\hat{Q}_{gc}$ and $\hat{Q}$ with $\hat{Q}^\dagger_{gc}$ and $\hat{Q}^\dagger$ respectively, the condition that $\bar{\psi}_{g+1}$ and $\bar{\psi}_{(-g-1)}$ be BRST closed

$$\hat{Q}^\dagger_{gc}\bar{\psi}_{g+1} = 0\,, \qquad \hat{Q}^\dagger\bar{\psi}_{g+1} \simeq 0\,, \tag{7.28}$$

$$\hat{Q}^\dagger_{gc}\bar{\psi}_{(-g-1)} = 0\,, \qquad \hat{Q}^\dagger\bar{\psi}_{(-g-1)} \simeq 0\,, \tag{7.29}$$

implies exactly the same equations of motion for A_α and $\tilde{A}^\alpha$ as before. Also, the BRST transformations of $\bar{\psi}_{g+1}$ and $\bar{\psi}_{(-g-1)}$ imply exactly the same gauge transformations of A_α and $\tilde{A}^\alpha$.

In the inner product, we choose the convention of initially placing $\bar{\psi}_{(-g-1)}$ on the left hand side, though we could just have easily chosen $\bar{\psi}_{g+1}$. Crucially, a generic BRST-exact operator, as in equation (4.8), obeys

$$\left(\bar{\psi}_{(-g-1)}, \{[\hat{U},\hat{Q}] + [\hat{U}_{gc},\hat{Q}_{gc}]\}\psi_{g+1}\right) = 0 \qquad \text{for } [\hat{U},\hat{Q}_{gc}] = 0\,, \tag{7.30}$$

using equations (4.9), (7.29) and the Jacobi identity. So we have seen that, by replacing ψ with $\bar{\psi}$ on the left of the inner product, the fact that $\hat{Q}$ and $\hat{Q}_{gc}$ aren't hermitian isn't problematic.

Let us calculate a general inner product,

$$\begin{aligned}(\bar{\psi}_{(-g-1)},\psi_{g+1}) &= \int \left[d^{10}X\, d^{16}\theta\, d^{16}\lambda\, d^{16}\bar{\lambda}\, d^{10}C\, d^{16}U \cdots\right] (\bar{\psi}_{(-g-1)})^*\psi_{g+1} \\ &= \int d^{10}X\, d^{16}\theta\, \tilde{A}^\alpha(X,\theta)A_\alpha(X,\theta).\end{aligned} \tag{7.31}$$

This tells us that on expanding $A_\alpha(X,\theta)$ in powers of θ^α, the coefficient of the $(\theta)^i$ term in $A_\alpha(X,\theta)$ couples to the coefficient of the $\theta^{(16-i)}$ term in $\tilde{A}^\alpha(X,\theta)$. Therefore just as we expand A_α in increasing powers of θ^α starting at 1 as in equation (E.14), so it makes sense to expand $\tilde{A}^\alpha$ in decreasing powers of θ^α starting with $\theta^1\theta^2\cdots\theta^{16}$. For this purpose, we invent a useful notation,

$$\tilde{\theta}_\alpha\tilde{\theta}_\beta\cdots = \frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\cdots(\theta^1\theta^2\cdots\theta^{16}) \tag{7.32}$$

where $\hat{\theta}^\alpha$ denotes that θ^α has been excluded from the product and the sign depends on α,β etc.. Also,

$$\theta^\rho\tilde{\theta}_\alpha\tilde{\theta}_\beta\cdots = \frac{\partial}{\partial\tilde{\theta}_\rho}\tilde{\theta}_\alpha\tilde{\theta}_\beta\cdots\,, \tag{7.33}$$

so that the covariant derivative can be written

$$D_\alpha \equiv \tilde{\theta}_\alpha - i\gamma^m_{\alpha\beta}\partial_m\frac{\partial}{\partial\tilde{\theta}_\beta}\,, \tag{7.34}$$

which is useful for making component calculations with $\tilde{A}^\alpha$. We observe that the gauge transformation of $\tilde{A}^\alpha$ does a similar job to the equation of motion for A_α and vice versa.

The superfield $\tilde{A}^\alpha$ has two physical components, $\tilde{a}^m(X)$ and $\tilde{\chi}_\alpha(X)$. We can choose a special gauge for $\tilde{A}^\alpha$ analogous to that for A_α in equation (E.14), such that

$$\tilde{A}^\alpha = i\tilde{a}^m(X)\gamma_m^{\alpha\beta}\tilde{\theta}_\beta - \tilde{\chi}_\gamma(X)\gamma_m^{\alpha\beta}\gamma^{m\gamma\delta}\tilde{\theta}_\beta\tilde{\theta}_\delta + \cdots\,, \tag{7.35}$$

and all remaining components depend only on $\tilde{a}^m$ and $\tilde{\chi}_\alpha$. The equation of motion (7.21) implies

$$\partial_m\tilde{a}^m(X) = 0\,, \tag{7.36}$$

and the gauge transformation (7.23) implies

$$\delta\tilde{a}^m = \partial_n(\partial^n s^m - \partial^m s^n)\,, \qquad \delta\tilde{\chi}_\alpha = \gamma^m_{\alpha\beta}\partial_m\xi^\beta\,, \tag{7.37}$$

for arbitrary parameters $s^n(X)$ and $\xi^\beta(X)$. Notice that the inner product, in the last line of equation (7.31), is gauge invariant with respect to variations in A_α due to the equations of motion of $\tilde{A}^\alpha$, and vice versa.

A derivation of the expression for (7.31) in terms of component fields has yet to be completed, due to the length of the calculation. Nevertheless, we deduce that up to normalization factors

$$(\bar{\psi}_{(-g-1)}, \psi_{g+1}) = \int d^{10}X\,(\tilde{a}^m(X)a_m(X) + \tilde{\chi}_\alpha(X)\chi^\alpha(X))\,, \tag{7.38}$$

for a number of reasons. Firstly, it must be gauge invariant and thus depend only on physical components. Secondly, since A_α and $\tilde{A}^\alpha$ are linear in their physical components, the inner product must also be linear in them. Thirdly, since a_m appears only at odd powers of θ^α, and χ^α only at even powers of θ^α in A_α, and since $\tilde{a}^m$ appears only at odd powers of $\tilde{\theta}_\alpha$ and $\tilde{\chi}_\alpha$ only at even powers of $\tilde{\theta}_\alpha$ in $\tilde{A}^\alpha$, so the inner product must be a sum of just two terms, one dependent only on a_m and $\tilde{a}^m$, the other only on χ^α and $\tilde{\chi}_\alpha$. The above expression is the only gauge invariant possibility which fits the above criteria.

In order to write down a basis for physical states ψ_{g+1}, we create states with definite quantum numbers, which are defined up to a BRST-exact wavefunction

$$\psi_{g+1} \sim \psi_{g+1}(k_m, a_m, \chi^\alpha)\,, \tag{7.39}$$

where k_m, a_m and χ^α are all constant, real numbers, such that

$$k^2 = 0\,, \qquad k^m a_m = 0\,, \qquad \gamma^m_{\alpha\beta}k_m\chi^\beta = 0\,. \tag{7.40}$$

Similarly,

$$\psi_{(-g-1)} \sim \psi_{(-g-1)}(k_m, \tilde{a}^m, \tilde{\chi}_\alpha)\,, \tag{7.41}$$

where k_m, $\tilde{a}^m$ and $\tilde{\chi}_\alpha$ are all constant, real numbers, such that

$$k^2 = 0\,, \qquad k_m\tilde{a}^m = 0\,. \tag{7.42}$$

7.3.3 Comparison with the light-cone gauge, BS superparticle

We relate a state $\psi_{g+1}(k_m, a_m, \chi^\alpha)$ to its light cone gauge, BS equivalent, by gauge-fixing $a^+ = 0$, as in appendix E.2.1. This will be important for comparing operators of the BS superparticle in the light-cone gauge with operators in our superparticle, by observing how they act on equivalent states. We write a generic wavefunction ψ_{g+1} with light-cone gauge values of k_m, a_m and χ^α

$$\psi_{g+1} \simeq \psi^{LC}_{g+1}(k_m, a_m, \chi^\alpha) \tag{7.43}$$

where $a^-(a^i) = (k^+)^{-1} k^i a^i$, and where we are using the standard light-cone gauge notation as in appendix E. Also, χ^b is determined as a function of $\chi^{\dot{b}}$ as in equation (E.10), since χ^α obeys the Dirac equation. This maps directly to the semi-light-cone gauge Brink-Schwarz wavefunction ψ_{BS}, with the usual notation as described in appendix E.1.

$$\psi_{BS} = \exp\left(ik_m X^m\right)\left(a^i|i\rangle + -i2^{\frac{1}{4}}(P^+)^{-\frac{1}{2}}\chi^{\dot{b}}|\dot{b}\rangle\right). \tag{7.44}$$

Likewise,

$$\bar{\psi}_{(-g-1)} \simeq \bar{\psi}^{LC}_{(-g-1)}(k_m, \tilde{a}^m, \tilde{\chi}_\alpha)\,, \tag{7.45}$$

where we fix the gauge symmetry of equation (7.37) with conditions $\tilde{a}^+ = 0$ and $\tilde{\chi}^b = 0$.

We compare our Schrödinger inner product to that of the semi-light cone gauge, BS superparticle. From equation (7.38), we learn

$$\left(\bar{\psi}^{LC}_{(-g-1)}, \psi^{LC}_{g+1}\right) = \int d^{10}X\left(\tilde{a}^i a^i + \tilde{\chi}^{\dot{b}}\chi^{\dot{b}}\right), \tag{7.46}$$

which agrees with the semi-light-cone gauge inner product in equation (E.4) up to a normalization factor. We therefore make the map between the two, isomorphic, state cohomologies

$$\psi^{LC}_{g+1}\left(k_m, (0, a^-(a^i), a^i),\, (\chi^b(\chi^{\dot{b}}), \chi^{\dot{b}}\right) \longleftrightarrow \bar{\psi}^{LC}_{(-g-1)}\left(k_m, (0, \tilde{a}^-(\tilde{a}^i), \tilde{a}^i), (0, \tilde{\chi}^{\dot{b}})\right), \tag{7.47}$$

in a similar manner to equation (B.4).

7.3.4 An analogy with abelian Chern-Simons theory

The manner in which the physical wavefunction obtained from the Berkovits BRST operator describes super Yang-Mills is unusual. In particular, the wavefunction $\lambda^\alpha A_\alpha$ appears at a ghost number one higher than that required to impose the constraints in $\hat{Q}$ as Dirac constraints. It was noticed however [2], that there is a more simple precedent. Witten [24] shows how Chern-Simons theory arises in a similar way from a string theory, which is easily modified to a particle theory. An analogy can be drawn between the Witten particle theory and the Berkovits superparticle theory. Our analogy differs significantly to that of Berkovits' [2].

Abelian, Chern-Simons theory can be described by the world-line, Witten action

$$S = \int d\tau\left(\dot{X}^m P_m - l^m P_m\right), \tag{7.48}$$

where $m = 0,1,2$, which is the hamiltonian form of the theory described by a zero lagrangian. First class constraints P_m imply the BRST operator

$$Q = -ic^m \partial_m \,, \tag{7.49}$$

where c^m, b_m are conjugate pairs of fermionic ghosts. The most general wavefunction can be expressed

$$\psi_W = C(X) + c^m A_m(X) + \frac{i}{2} c^m c^n \varepsilon_{mnp} A^{*p}(X) + \frac{i}{6} c^m c^n c^p \varepsilon_{mnp} C^*(X) \,, \tag{7.50}$$

which terminates because c^m is fermionic. The condition $Q\psi = 0$, together with the BRST transformation $\delta\psi = Q\Omega(c,X)$ for the particle model, imply the equations of motion and gauge transformations for the Chern-Simons fields

$$\partial_{[m} A_{n]} = 0 \,, \qquad \delta A_m = \partial_m \Lambda, \tag{7.51}$$

$$\partial_p A^{*p} = 0 \,, \qquad \delta A^{*p} = \varepsilon^{pmn} \partial_m w_n, \tag{7.52}$$

$$\partial_m C = 0 \,, \qquad \delta C^* = \partial_p u^p \,. \tag{7.53}$$

The Chern-Simons action is given by

$$\int d^3X \left(\frac{1}{2} \varepsilon^{mnp} A_m \partial_n A_p + i A^{*p} \partial_p C \right) , \tag{7.54}$$

where A^{*p} is the antifield to A_p and C and C^* are the ghost and anti-ghost, and can be written remarkably compactly as

$$S = \frac{1}{2} (\psi \,, Q\,\psi) \,, \tag{7.55}$$

where the inner product measure is the usual $d^3X d^3c$.

To make the analogy clear, we rewrite the Witten wavefunction for the corresponding particle theory in the form

$$\psi_W = C(X)\phi_{b=0} + A_m(X)\hat{c}^m \phi_{b=0} + iA^{*m}(X)\hat{b}_m \phi_{c=0} + iC^*(X)\phi_{c=0} \,, \tag{7.56}$$

where $\hat{G}\phi_{b=0} = -3/2\phi_{b=0}$ and $\hat{G}\phi_{c=0} = 3/2\phi_{c=0}$. The point to notice is that fields couple to anti-fields in the Schrödinger inner product (ψ_W, ψ_W), since $(\phi_{b=0}, \phi_{c=0}) = 1$ and $(\hat{c}^m \phi_{b=0}, \hat{b}_n \phi_{c=0}) = -i\delta^m_n$, and all other inner products are zero. In other words, states at opposite ghost number, as described by anti-hermitian ghost number operator $\hat{G}$, couple to each other, as in equation (B.1). Furthermore, the state cohomology $H^g_{st}(\hat{Q})$ is dual to $H^{-g}_{st}(\hat{Q})$, and as part of their special relation, the equation stating that $\psi_g \in H^g_{st}(\hat{Q})$ is BRST-closed is connected in a specific way to the equation that states the BRST transformation for $\psi_{-g} \in H^{-g}_{st}(\hat{Q})$, and vice versa. This is the reason why fields and anti-fields appear at opposite ghost number, since the equations of motion of a field are related in just the right way to the gauge invariances of the corresponding antifield, and vice versa.

Our wavefunction for the superparticle is thus

$$\psi = C(X,\theta)\phi_g + A_\alpha(X,\theta)\hat{\lambda}^\alpha \phi_g + A^{*\alpha}(X,\theta)\hat{w}_\alpha \phi_{-g} + C^*(X,\theta)\phi_{-g} \,, \tag{7.57}$$

where $A^{*\alpha}(X,\theta)$ is the super-antifield to $A_\alpha(X,\theta)$ and $C^*(X,\theta)$ the super-antighost to super-ghost $C(X,\theta)$ and where we recall that $\phi_g = \phi_{w=0,\ldots}$ and $\phi_{-g} = \phi_{\lambda=0,\ldots}$.

We might expect to be able to write a BV, superspace action for super Maxwell in an analogous way to the Chern-Simons action in equation (7.55), however the action

$$S = (\bar{\psi}, \hat{Q}\psi) = \int d^{10}X d^{16}\theta \, A^{*\alpha}(X,\theta) D_\alpha C(X,\theta) \,, \tag{7.58}$$

unfortunately only provides the BV super-ghost, super-anti-field part. This is because we need a ghost number $-(g+2)$ term in ψ in order to couple to the term $\hat{Q}A_\alpha(X,\theta)\hat{\lambda}^\alpha \phi_g$. Of course this should come as no surprise, since there is no known action principle for the superspace formulation of $D = 10$ super Yang-Mills. The superfield ghosts and anti-field are only realized on-shell.

A different analogy to Chern-Simons theory is also drawn by Berkovits [2]. The main difference is that he chooses a particular non-linear measure, similar to that used in his expression for massless tree-level amplitudes, instead of the natural measure used here. The principle is still the same, that C must couple to C^*, and A to A^* under the inner product. However, due to the form of the measure, A^* and C^* appear at ghost numbers $g+2$ and $g+3$ respectively, though with extra indices i.e. $A^{*\alpha\beta}$ and $C^{*\alpha\beta\gamma}$. It is claimed however that their on-shell physical components still correspond with the BV anti-fields of super-Maxwell.

7.4 The physical operators

7.4.1 The pure spinor, operator cohomology $H^0_{op}(\hat{Q}_{gc})$

Due to the fact that no completion of $\hat{Q}_{gc}$ has been found yet, we cannot calculate all operators belonging to $H^0_{op}(\hat{Q}_{gc})$ explicitly. However, if we further restrict ourselves to operators which have zero ghost numbers, as do physical operators, we can specify a basis for all cohomology classes. Furthermore we can deduce the algebra of the basis elements, which is closed, and how the basis elements act on physical states.

The operators $\hat{\lambda}^\alpha \hat{w}_\alpha$ and $\hat{\lambda}^\alpha {\gamma^{mn}}_\alpha{}^\beta \hat{w}_\beta/2$ form a basis for ghost number zero operators, which are gauge invariant with respect to the first class, ghost constraints. A general such gauge invariant operator is of the form

$$\hat{F} =: \hat{F}_0\left(\hat{\lambda}^\alpha \hat{w}_\alpha, \frac{1}{2}\hat{\lambda}\gamma^{mn}\hat{w}\right) : + : \hat{F}_{1m}(\hat{\lambda},\hat{w})\hat{\lambda}\gamma^m\hat{\lambda} : , \tag{7.59}$$

for some convenient normal ordering, where F_{1m} has ghost number -2, but is otherwise arbitrary and where the second term vanishes on the constraint surface.

The BRST extensions of the basis elements are $\hat{\lambda}^\alpha \hat{w}_\alpha + \hat{E}$ and $\hat{\lambda}\gamma^{mn}\hat{w}/2 + \hat{L}^{mn}$, where

$$\hat{E} = 2\hat{B}^m \hat{C}_m - 3\hat{U}^\alpha \hat{V}_\alpha + 4\hat{B}_{np}\hat{C}^{np} - \cdots \,, \tag{7.60}$$

$$\hat{L}^{mn} = \hat{C}^m \hat{B}^n - \hat{C}^n \hat{B}^m + \frac{1}{2}\hat{U}^\alpha {(\gamma^{mn})_\alpha}^\beta \hat{V}_\beta + \cdots \,. \tag{7.61}$$

We notice that $\hat{\lambda}\gamma^{mn}\hat{w}/2$ generates Lorentz transformations for $\hat{\lambda}^\alpha$ and $\hat{w}_\alpha$, and hence the equation $[\hat{\lambda}\gamma^{mn}\hat{w}/2 + \hat{L}^{mn}, \hat{Q}_{gc}] = 0$ implies that $\hat{L}^{mn}$ is the Lorentz generator for the ghost constraint ghosts $\hat{C}$, $\hat{B}$, ..., if we assume $\hat{Q}_{gc}$ to be a Lorentz scalar. Therefore, the algebra of $\hat{E}$ and $\hat{L}^{mn}$ is given by

$$[\hat{E}, \hat{L}^{mn}] = 0 \qquad [\hat{L}^{mn}, \hat{L}^{pq}] = i(\eta^{np}\hat{L}^{mq} - \eta^{mp}\hat{L}^{nq} + \eta^{mq}\hat{L}^{np} - \eta^{nq}\hat{L}^{mp})\,. \tag{7.62}$$

The scheme for constructing the BRST extension of any gauge invariant operator, is firstly to split it into the form of equation (7.59) and discard the piece proportional to $\hat{\lambda}\gamma^m\hat{\lambda}$, whose BRST extension is always $\hat{Q}_{gc}$-exact. We then replace $\hat{\lambda}^\alpha\hat{w}_\alpha$ and $\hat{\lambda}^\alpha\gamma^{mn}{}_\alpha{}^\beta\hat{w}_\beta/2$ in $\hat{F}_0$ with their BRST extensions $(\hat{\lambda}^\alpha\hat{w}_\alpha + \hat{E})$ and $(\hat{\lambda}^\alpha\gamma^{mn}{}_\alpha{}^\beta\hat{w}_\beta/2 + \hat{L}^{mn})$.

We deduce that $\hat{L}^{mn}$ annihilates physical states due to being antisymmetric in ghost constraint ghosts, and $\hat{E}\phi_g = 0$, but $\hat{E}\phi_{-g} = -i(5+k)\phi_{-g}$, where recall that k is an unknown constant. If necessary, the unknown constant $-i(5+k)$ can be subtracted from $\hat{E}$ to begin with, or when calculating expectation values, we may place the negative ghost number wavefunction on the left hand side of the inner product.

By following the above scheme, we can perform matrix element calculations and compute quantum brackets of physical operators etc.. The price to be paid is that, each cohomology class of $H^0_{op}(\hat{Q}_{gc})$ has only one representative using our basis.

7.4.2 The physical operator cohomology $H^0_{op}(\hat{Q}|H^0_{op}(\hat{Q}_{gc}))$ modulo 'effective constraints'

A basis for operators belonging to $H^0_{op}(\hat{Q}|H^0_{op}(\hat{Q}_{gc}))$, linear in phase-space variables, is given by

$$\hat{P}_m, \hat{q}_\alpha, \hat{K}^{mn}, \hat{J}\,, \tag{7.63}$$

where

$$\hat{q}_\alpha = \hat{p}_\alpha + i\hat{P}_m(\gamma^m\hat{\theta})_\alpha\,, \tag{7.64}$$

$$\hat{K}^{mn} = \hat{X}^m\hat{P}^n - \hat{X}^n\hat{P}^m + \frac{1}{2}\hat{\theta}^\alpha(\gamma^{mn})_\alpha{}^\beta\hat{p}_\beta + \frac{1}{2}\hat{\lambda}^\alpha(\gamma^{mn})_\alpha{}^\beta\hat{w}_\beta + \hat{L}^{mn}\,, \tag{7.65}$$

$$\hat{J} = 2\hat{X}^m\hat{P}_m + \hat{\theta}^\alpha\hat{p}_\alpha + \hat{\lambda}^\alpha\hat{w}_\alpha + \hat{E}\,. \tag{7.66}$$

Any $\hat{Q}_{gc}$-closed, ghost number -1 operator $\hat{A} \in H^0_{op}(\hat{Q}_{gc})$ is also $\hat{Q}_{gc}$-exact, because its gauge invariant piece $\hat{A}_0(\hat{\lambda}, \hat{w})$ must be proportional to $\hat{\lambda}\gamma^m\hat{\lambda}$. Thus, $[\hat{A}, \hat{Q}] \simeq 0$, meaning that there are no $\hat{Q}$-exact operators, which aren't trivial, i.e. $\hat{Q}_{gc}$-exact.

The operator cohomology $H^0_{op}(\hat{Q}|H^0_{op}(\hat{Q}_{gc}))$ cannot correspond with the light-cone gauge space of operators for the BS superparticle for two reasons. There is no mass-shell constraint $\hat{P}^2 = 0$, which would render $\hat{P}^2$ BRST exact, and there are 8 too many independent fermionic operators $\hat{q}_\alpha$, compared to the 8 $\hat{\theta}^a$'s of the light-cone gauge BS superparticle. However, something interesting happens, which saves us. We find that the matrix element of $\hat{P}^2$ between arbitrary physical states $(\bar{\psi}_{(-g-1)}, \hat{P}^2\psi_{g+1})$, which we denote as $\langle\hat{P}^2\rangle$, obeys,

$$\langle\hat{P}^2\rangle = i(\bar{\psi}_{(-g-1)}, \hat{Q}\partial^m A_m(X,\theta)\phi_g) = 0\,, \tag{7.67}$$

where we have used the super-Maxwell field equation $\partial^m F_{m\alpha} = 0$, where $F_{m\alpha}$ is the spin 3/2 super field-strength and A_m the space-time super gauge connection, as in appendix E.2.2.

This perhaps isn't so surprising, since the striking feature of super-Maxwell in ten dimensions is that the constraint equations alone place the theory on-shell. Also,

$$\langle \hat{P}_m \gamma^{m\alpha\beta} \hat{q}_\beta \rangle = -4(\bar{\psi}_{(-g-1)}, \hat{Q} W^\alpha \phi_g) = 0 \tag{7.68}$$

where we use the abelian form of the constraint equation (E.11), the field equation $\partial^m F_{m\alpha} = 0$ and the identity $D_\beta W^\alpha = F_{mn} (\gamma^{mn})^\alpha{}_\beta W^\beta / 2$, where $W^\beta = \gamma^{m\alpha\beta} F_{m\alpha}$ is the photino superfield strength. Since $\hat{q}_\alpha$ obeys the Dirac equation in (7.68), it effectively has the required 8 independent degrees of freedom.

We describe $\hat{P}^2 = 0$ and $\hat{P}_m \gamma^{m\alpha\beta} \hat{q}_\beta = 0$ as 'effective constraints', since they arise only indirectly from the Berkovits BRST operator $\hat{Q}$. All other 'effective constraints' are formed from these two expressions.

Given a generic effective constraint $\hat{G}_{\text{eff}}$, we deduce that

$$\langle \hat{G}_{\text{eff}} \hat{A} \rangle = \langle \hat{A} \hat{G}_{\text{eff}} \rangle = 0 \qquad \text{iff } \hat{A} \in H^0_{op}(\hat{Q} | H^0_{op}(\hat{Q}_{gc}))\,. \tag{7.69}$$

An interesting inference is that

$$[\hat{A}, \hat{P}^2] \approx 0\,, \qquad [\hat{A}, \hat{P}_m \gamma^{m\alpha\beta} \hat{q}_\beta] \approx 0\,, \tag{7.70}$$

given $\hat{A} \in H^0_{op}(\hat{Q} | H^0_{op}(\hat{Q}_{gc}))$, where the $\approx$ refers to the effective constraint surface. We observe that the effective constraint surface is the same as the first class part of the BS superparticle constraint surface, which describes Siegel's superparticle model. We have now completed the superparticle model, with the exception of not having produced an explicit completion of the pure spinor BRST operator $\hat{Q}_{gc}$, which as argued in section 7.4.1, is not as restrictive as one might expect.

It seems plausible that $H^0_{op}(\hat{Q} | H^0_{op}(\hat{Q}_{gc}))$ modulo the effective constraints, corresponds with the space of light-cone gauge, BS operators. In the next subsection, we explicitly construct the map from the light-cone gauge BS operators to our 'physical' operators.

7.4.3 Comparison with the light-cone gauge, BS superparticle

Since we can map between any state $\psi_{g+1}(k_m, a_m, \chi^\alpha)$ in our model and the corresponding state $\psi_{BS}(k_m, a^i, \chi^{\dot{a}})$ in the light-cone gauge, BS superparticle as in section 7.3.3, we can also relate operators in the two models, which can be defined by how they act on the states. We attempt to map the physical operators of the light-cone gauge, BS superparticle to our basis operators in equation (7.63).

Our $\hat{P}_m$'s, combined with effective constraint $\langle \hat{P}^2 \rangle = 0$, straightforwardly map to the $\hat{P}_m$'s of the BS model. The fermionic operators $\hat{q}_\alpha$ of our model, in the Schrödinger representation, are simply the supersymmetry generators

$$Q_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} + i \gamma^m_{\alpha\beta} \theta^\beta \partial_m\,. \tag{7.71}$$

As a result of the effective constraint (7.68), we can write $Q^{\dot{a}}$ in terms of Q^a,

$$\langle Q^{\dot{a}} \rangle = \left\langle 2^{-\frac{1}{2}} (\hat{P}^+)^{-1} \hat{P}^i \sigma^i_{\dot{a}a} Q^a \right\rangle, \tag{7.72}$$

where $Q_\alpha = (Q^a, Q^{\dot a})$, and $\sigma^i_{\dot a a}$ are the SO(8) gamma matrices defined in appendix C. Therefore, $Q^{\dot a}$ is redundant.

Let us see how Q^a behaves by observing how it acts on a generic state ψ_{g+1}. We firstly choose a representative physical state from each cohomology class, with light-cone gauge quantum numbers as in section 7.3.3

$$\psi^{LC}_{g+1} = \psi_{g+1}\left(k_m, (0, a^-(a^i), a^i), (\chi^a(\chi^{\dot a}), \chi^{\dot a})\right), \tag{7.73}$$

and now calculate

$$Q_\alpha \psi_{g+1}((k_m, a_m, \chi^\beta)) \sim \psi_{g+1}\left(k_m, (\gamma_m \chi)_\alpha, -k_m a_n (\gamma^{mn})_\alpha{}^\beta\right). \tag{7.74}$$

We can obtain this either with a calculation of $Q_\alpha A_\beta$ in components, or more simply, by reading off the super-Maxwell, supersymmetry transformations of equation (E.8) up to a factor, since Q_α are also the super-Maxwell supersymmetry generators. Combining the above two equations, we learn

$$Q^a \psi^{LC}_{g+1} \sim \psi^{LC}_{g+1}\left(k_m, (0, a^{-\prime}(a^{i\prime}), a^{i\prime}), (\chi^{a\prime}(\chi^{\dot a\prime}), \chi^{\dot a\prime})\right), \tag{7.75}$$

where

$$a^{i\prime} = \sigma^i_{a\dot b}\chi^{\dot b} \qquad \chi^{\dot a\prime} = -(2)^{\frac12} k^+ a^i \sigma^i_{a\dot b}\,. \tag{7.76}$$

Therefore, using the map between ψ^{LC}_{g+1} and ψ_{BS} in section 7.3.3, we make the relation

$$Q^a \equiv i 2^{\frac14} (P^+)^{\frac12} S^a\,, \tag{7.77}$$

where S^a describe the fermionic degrees of freedom for the light-cone gauge, BS superparticle and where we have used equation (E.3).

The mapping between the $\hat X$'s of the two models is more involved, so we simply provide the outline of a proof. To begin with we relate our $\hat K^{mn}$, defined in equation (7.65), to gauge-invariant $(\hat X^m \hat P^n - \hat X^n \hat P^m)$ of the semi-light-cone gauge, BS model. The $(\hat X^m \hat P^n - \hat X^n \hat P^m)$ part of $\hat K^{mn}$ operates on ψ^{LC}_{g+1} in an identical manner to how it operates on ψ_{BS}. Unfortunately, however, all the terms in $\hat K^{mn}$ are necessary in order that it be BRST closed. When $\hat K^{mn}$ operates on a physical state $\psi_{g+1}(k_m, a_m, \chi^\alpha)$, it Lorentz rotates the quantum numbers. We observe that $(\hat X^m \hat P^n - \hat X^n \hat P^m)$ Lorentz rotates k_m, while the remaining terms in $\hat K^{mn}$ Lorentz rotate a_m and χ^α. We can build an operator $\hat S^{mn}$ out of $\hat q_\alpha$'s, which compensates for the rotation of a_m and χ^α, without rotating k_m. The SO(8) part of this term, for example, would be $\hat S^{ij} \sim \hat S^a \sigma^{ij}_{ab} \hat S^b$, where $\hat S^a$ has been defined in terms of Q^a in equation (7.77). Then

$$\hat R^{mn} = \hat K^{mn} + \hat S^{mn}(\hat q_\alpha) \tag{7.78}$$

is exactly equivalent to $(\hat X^m \hat P^n - \hat X^n \hat P^m)$ of the semi-light-cone, BS model.

It is fairly straightforward to relate $\hat X^i$ and $\hat X^-$ of the light-cone gauge, BS model with $\hat R^{mn}$ of our model. We first relate $\hat X^i$ and $\hat X^-$ to their gauge-invariant counterparts

$$\text{light-cone gauge BS} \longleftrightarrow \text{gauge invariant with respect to } P^2 = 0 \tag{7.79}$$

$$X^i \longleftrightarrow X^i - (P^+)^{-1}P^i(X^+ - \tau P^+) \tag{7.80}$$
$$X^- \longleftrightarrow X^- - (P^+)^{-1}P^-(X^+ - \tau P^+), \tag{7.81}$$

where the expressions on the left and right hand side are equal on the light-cone gauge constraint surface. A convenient basis for these operators is

$$\hat{P}^i, \hat{P}^+, (\hat{X}^i\hat{P}^+ - \hat{X}^+\hat{P}^i), (\hat{X}^-\hat{P}^+ - \hat{X}^+\hat{P}^-)\,. \tag{7.82}$$

Thus, any light-cone gauge, BS operator can be mapped to an operator in our model, formed of the following basis elements

$$\hat{P}_m, \hat{q}^a, \hat{R}^{i+}, \hat{R}^{-+}\,. \tag{7.83}$$

To prove the reverse mapping for the $\hat{X}$'s is more difficult, though given equation (7.70), it seems reasonable to conjecture that every operator in our model can be mapped to an operator in the light-cone gauge, BS superparticle.

8. Central charge cancellation for the open superstring

In principle, the methods used in quantizing the superparticle here can also be generalized to quantize the free superstring. It is further confirmation of the fundamental nature of the Berkovits BRST operator, combined with pure spinor ghosts, that the first, excited massive, superspace vertex operator [10] has been explicitly constructed, providing for the first time the superspace form of the first massive multiplet. Furthermore, the same principles can be used to covariantly obtain the rest of the physical spectrum.

There are additional issues with the superstring which don't apply to the superparticle. In particular, there is a quantum anomaly which is the central charge in the Virasoro algebra. One expects the central charge to disappear in $D = 10$ as with the RNS superstring.

The BRST charges are now

$$Q = \oint dz \lambda^\alpha(z) d_\alpha(z)\,, \qquad Q_{gc} = \oint dz (C_m \lambda\gamma^m\lambda + \cdots)\,, \tag{8.1}$$

where we use the same notation as Berkovits [4], thus simply replacing world-line parameter τ with complex, euclidean world-sheet parameter z.

The left-moving part of the superstring action is defined as

$$S = \int d^2z \left(\frac{1}{2}\partial X_m \bar{\partial} X^m + \bar{\partial}\theta^\alpha p_\alpha + \bar{\partial}\lambda^\alpha w_\alpha + \bar{\partial} C_m B^m + \bar{\partial} U^\alpha V_\alpha + \cdots\right), \tag{8.2}$$

which is the open superstring version of equation (7.6). Hence, the energy momentum tensor is given by

$$T_{zz}(z) = \frac{1}{2}\partial X_m \partial X^m + \partial\theta^\alpha p_\alpha + \partial\lambda^\alpha w_\alpha + \partial C_m B^m + \partial U^\alpha V_\alpha + \cdots\,. \tag{8.3}$$

The central charge contributions from X, (p, θ) and (w, λ) are $+10$, -32, $+32$ respectively and from the ghost pairs (B, C), (V, U), ... are -20, $+32$, ... Each fermionic ghost pair contributes -2 and each bosonic pair $+2$. From equation (6.6), the graded sum of ghost constraint ghost degrees of freedom, starting from $i = 1$ instead of $i = 0$, is -5. Thus, the total contribution to the central charge by the capital letter ghosts is $2 \times (-5) = -10$. The total central charge is then

$$c = 10 - 32 + 32 - 10 = 0\,, \tag{8.4}$$

as required, assuming that a termination for Q_{gc} can be found. If there are infinite ghosts for ghosts, c will be an infinite sum which must be regularized.

9. Future research

Either the ghosts for ghosts terms in $\hat{Q}_{gc}$ have to be completed, or some other method used before the ten dimensional pure spinor and its conjugate momentum are covariantly quantized. Only then will we have a complete, covariant BRST system for the Berkovits superparticle.

Despite the above problem, we have seen in section 7.4.1, how it's still possible to covariantly calculate matrix elements of arbitrary physical operators, between physical states. Therefore it seems logical to continue with the next step and build a model, in the same vein as this paper, for the free Berkovits superstring, in the hope that we can still perform useful calculations.

An outstanding problem is to derive tree-level superstring scattering amplitudes. Although a plausible expression for massless tree amplitudes has been conjectured, and tested [4, 5], it uses a special integration measure, whose precise origin is unknown. Understanding the origin of the tree-level amplitudes seems a necessary step before we have a realistic chance of obtaining one-loop amplitudes. It is hoped that understanding how to write free superstring matrix elements, in a similar manner to the superparticle in this article, will provide some insight towards this goal. After all, in general we construct interacting string amplitudes using the corresponding free string model.

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A. Conventions

Roman letters in the middle of the alphabet m, n, p etc. correspond to space-time indices. Greek letters at the start of the alphabet are used as spinor indices. The flat space-time metric η^{mn} has signature $- + + \ldots +$. We also choose units such that $c = 1$ and $\hbar = 1$.

Throughout, the graded Poisson bracket of functions A and B is given by

$$[A, B]. \tag{A.1}$$

In the context of operators, which have hats,

$$[\hat{A}, \hat{B}], \tag{A.2}$$

is the graded quantum (anti-)commutator of operators $\hat{A}$ and $\hat{B}$. The brackets of generic, bosonic, conjugate pair $\hat{X}$ and $\hat{P}$, and generic, fermionic, conjugate pair $\hat{C}$ and $\hat{B}$ are given by

$$[\hat{X}, \hat{P}] = i\,, \qquad [\hat{C}, \hat{B}] = -i\,. \tag{A.3}$$

Also $\hat{X}$, $\hat{P}$ and $\hat{C}$ are hermitian, and $\hat{B}$ is anti-hermitian. In the Schrödinger representation, $\hat{X}$ and $\hat{C}$ are simply given by bosonic variable X and fermionic variable C respectively. Similarly, their conjugate momenta $\hat{P}$ and $\hat{B}$ are given by $-i\partial/\partial X$ and $-i\partial/\partial C$.

B. BRST quantization in the Schrödinger representation

We state some useful results [18] regarding BRST quantization in the Schrödinger representation.

The ghost number operator $\hat{G}$ is defined up to a constant, which can be chosen such that it is anti-hermitian $\hat{G} = -\hat{G}^\dagger$. We then find that the inner product of two states ψ_g and $\psi_{g'}$ obeys

$$(\psi_g, \psi_{g'}) = 0\,, \qquad \text{for } g + g' \neq 0\,, \tag{B.1}$$

where $\hat{G}\psi_g = g\psi_g$ and $\hat{G}\psi_{g'} = g'\psi_g$. So for $g \neq 0$, the state ψ_g has zero norm and couples only to states with ghost number $-g$. Also, there is a theorem that opposite ghost number, state cohomologies are isomorphic

$$H^g_{st}(\hat{Q}) \cong H^{-g}_{st}(\hat{Q})\,, \tag{B.2}$$

where $\hat{Q}$ is the BRST operator.

In the Schrödinger representation, with no non-minimal sector included, the physical state cohomology appears at ghost numbers $\pm m/2$ for a standard gauge theory with m irreducible, first class constraints.

We therefore need to compute state cohomologies at both ghost numbers in order to make matrix element calculations. They then take the form $(\chi_{gc}, \hat{A}\psi_{-g})$, where $\psi_{-g} \in H^{-g}(\hat{Q})$ and $\chi_{gc} \in H^g(\hat{Q})$ and where $\hat{A}$ is a ghost number zero operator. For practical calculations, states in each cohomology will be defined by a different, but equivalent set of quantum numbers, which we therefore need to relate. We want an explicit map between cohomology classes at the two ghost numbers.

We look for a basis $\{\psi^A_g\}$ for states in $H^g_{st}(\hat{Q})$, and similarly a basis $\{\psi^B_{-g}\}$ for states in $H^{-g}_{st}(\hat{Q})$, where A and B are indices, such that

$$(\psi^B_{-g}, \psi^A_g) = \delta^{AB}\,, \tag{B.3}$$

and each cohomology class has just one representative which is a linear combination of the basis elements. We then make the map

$$\psi^A_g \longleftrightarrow \psi^A_{-g}\,. \tag{B.4}$$

C. $D = 10$ Gamma matrices

C.1 Construction and basic properties

The Dirac gamma matrices in ten dimensions are 32×32 matrices Γ^m_{AB} obeying the Clifford algebra

$$\Gamma^m_{AB}\Gamma^n_{BC} + \Gamma^n_{AB}\Gamma^m_{BC} = 2\eta^{mn}\delta_{AC}\,. \tag{C.1}$$

We choose the reducible, Majorana-Weyl representation, in which Γ^m_{AB}'s are real and consist of two symmetric 16×16 matrices $\gamma^m_{\alpha\beta}$ and $\gamma^{m\alpha\beta}$ on the off-diagonals

$$\Gamma^m = \begin{pmatrix} 0 & \gamma^{m\alpha\beta} \\ \gamma^m_{\alpha\beta} & 0 \end{pmatrix}. \tag{C.2}$$

In this notation, θ^α is Weyl and θ_α anti-Weyl, thus a down spinor index can only be contracted with an up index when building Lorentz covariant tensors. Since γ^m are real, the Majorana condition simply says that $\theta^\alpha = \theta^{\alpha *}$. We generally deal with only Weyl spinors and hence use the 16×16 γ^m notation.

The Clifford algebra in terms of γ^m reads

$$\gamma^m_{\alpha\beta}\gamma^{n\beta\delta} + \gamma^n_{\alpha\beta}\gamma^{m\beta\delta} = 2\eta^{mn}\delta^\delta_\alpha\,. \tag{C.3}$$

The 16×16 gamma matrices can be built from the SO(8) gamma matrices which themselves are direct products of Pauli matrices [25]. The antisymmetric, real SO(8) Pauli matrices $\{\sigma^i_{a\dot b}\, i = 1,\ldots,8\}$, which obey the Clifford algebra

$$\sigma^i_{a\dot a}\sigma^j_{\dot a b} + \sigma^j_{a\dot a}\sigma^i_{\dot a b} = 2\delta^{ij}\delta_{ab}\,, \tag{C.4}$$

can be used to construct $\gamma^m_{\alpha\beta}$. Specifically

$$\gamma^i_{\alpha\beta} = \begin{pmatrix} 0 & \sigma^i_{a\dot a} \\ \sigma^i_{\dot b b} & 0 \end{pmatrix}, \tag{C.5}$$

where $i = 1,\ldots,8$. We define $\gamma^{m\alpha\beta}$ by exactly the same expression. We see that a Weyl spinor splits as $\theta^\alpha = (\theta^a, \theta^{\dot a})$. A ninth matrix which anticommutes with these eight is given by $\gamma^9_{\alpha\beta} = \gamma^1_{\alpha\beta_1}\gamma^{2\beta_1\beta_2}\cdots\gamma^8_{\beta_7\beta}$, which given the SO(8) matrices we can calculate below. The values of $\gamma^0_{\alpha\beta}$ and $\gamma^{0\alpha\beta}$ are similarly defined in order to be consistent with their algebra (C.3)

$$\gamma^9_{\alpha\beta} = \gamma^{9\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.6}$$

$$\gamma^0_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^{0\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.7}$$

A generic antisymmetric product of r γ^m's is notated as

$$\gamma^{m_1 m_2\cdots m_r} = \gamma^{[m_1}\gamma^{m_2}\cdots\gamma^{m_r]}\,, \tag{C.8}$$

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where a factor of $1/r!$ is implicit, remembering that a $\gamma^{m_1}_{\alpha\beta}$ must contract with a $\gamma^{m_2\beta\delta}$ etc.. This larger set of gamma matrices, defined by the full set of antisymmetric combinations, form a basis for bispinors. There is a duality

$$\gamma^{m_1 m_2 \cdots m_r} = \frac{1}{(10-r)!}\varepsilon^{m_1\cdots m_r m_{r+1}\cdots m_{10}}\gamma_{m_{r+1}}\cdots\gamma_{m_{10}}\,, \tag{C.9}$$

in particular, γ^{mnpqr} is self-dual, so only half of the γ^{mnpqr}'s are independent. A generic bispinor with either 2 lower or 2 upper indices is a linear combination of γ^m, γ^{mnp} and γ^{mnpqr}. For example

$$f_{\alpha\beta} = f_m\gamma^m_{\alpha\beta} + f_{mnp}\gamma^{mnp}_{\alpha\beta} + f_{mnpqr}\gamma^{mnpqr}_{\alpha\beta}\,, \tag{C.10}$$

where f_m, f_{mnp} and f_{mnpqr} are calculated in terms of $f_{\alpha\beta}$ by using the orthogonal properties of the gamma matrices. For example $f_m = \gamma_m^{\alpha\beta} f_{\alpha\beta}/16$.

From the definition, γ^m and γ^{mnpqr} are symmetric, while γ^{mnp} is antisymmetric. Similarly, $\delta^\alpha{}_\beta$, $\gamma^{mn\alpha}{}_\beta$ and $\gamma^{mnpq\alpha}{}_\beta$ form a basis for bispinors with one lower and one upper index. The matrices $\delta^\alpha{}_\beta$, $\gamma^{mnpq\alpha}{}_\beta$ are symmetric, while $\gamma^{mn\alpha}{}_\beta$ is antisymmetric.

C.2 Gamma matrix identities

In principle, all identities can be calculated from the Fierz identity and the Clifford algebra identity given below

$$\gamma_{m\alpha(\beta}\gamma^m_{\gamma\delta)} = 0\,, \qquad \gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\delta} = \eta^{mn}\delta^\delta_\alpha\,, \tag{C.11}$$

though in practice, this is far too time consuming for all but the most simple identities. A common requirement is to calculate a product of γ's in terms of a sum of γ's. For this purpose, a slicker method is to use Young tableaux, which are useful for determining direct products of tensors in terms of sums of tensors with definite (anti)-symmetry properties in their indices. For example

$$\gamma^m\gamma^{npqr} = k_1\gamma^{mnpqr} + k_2\eta^{m[n}\gamma^{pqr]}\,, \tag{C.12}$$

where k_1 and k_2 are constants, and we have used that $\gamma^{(m}\gamma^{n)} = \eta^{mn}$. We then calculate k_1 and k_2 by substituting particular values of m,n,p,q,r into the above equation. In this case $k_1 = 1$ and $k_2 = 4$. Some more useful identities are

$$\gamma^{m\alpha\beta}\gamma^n_{\alpha\beta} = 16\eta^{mn}\,, \qquad \gamma^m_{\alpha\beta}\gamma_m^{\beta\delta} = 10\delta^\delta_\alpha\,, \tag{C.13}$$
$$\gamma^m\gamma^{np} = \gamma^{mnp} + 2\eta^{m[n}\gamma^{p]}\,. \tag{C.14}$$

D. Description of pure spinors using $\mathrm{U}(5)$ co-ordinates

By first Wick-rotating from $\mathrm{SO}(9,1)$ to $\mathrm{SO}(10)$ and using $\mathrm{U}(5)$ co-ordinates, we can parameterize the pure spinor constraint surface non-degenerately in a certain co-ordinate patch.

Using Berkovits' notation,

$$X^a = (X^1 + iX^2), \ldots, (X^9 + iX^{10}), \qquad a = 1, \ldots, 5, \tag{D.1}$$

$$X_a = X^{a\dagger} = (X^1 - iX^2), \ldots, (X^9 - iX^{10}), \tag{D.2}$$

where $X^{10} = -iX^0$. So X^a and X_a transform in the $\mathbf{5}$ and $\bar{\mathbf{5}}$ representation of the U(5) group. Thus, we define the U(5) gamma matrices γ^a and γ_a in the same manner, except we include a normalization factor so that $\gamma^a = (\gamma^1 + i\gamma^2)/\sqrt{2}$. The gamma matrix algebra is

$$\{\gamma^a, \gamma^b\} = \{\gamma_a, \gamma_b\} = 0, \qquad \{\gamma^a, \gamma_b\} = 2\delta^a_b, \tag{D.3}$$

so we can treat γ^a as raising and γ_a as lowering operators in order to create a generic spinor. For example, the ground state spinor u^α_+ is defined by $\gamma_a u_+ = 0$ for $a = 1, \ldots, 5$. By acting with up to 5 γ^a's on the ground state u_+, we obtain the full set of spinors. We see that acting with an odd number of γ^a's on u^α_+ changes the chirality, since an up α index can only contract with a down α index. For example u^α_+ and $\gamma^a_{\alpha\beta} u^\beta_+$ have opposite chirality. A basis for the spinor λ^α is given by u_+, $(u^{ab})^\alpha = \gamma^a\gamma^b u_+$ and $u^\alpha_a = \varepsilon_{abcde}\gamma^b\gamma^c\gamma^d\gamma^e u_+$. Thus

$$\lambda^\alpha = \lambda^+ u^\alpha_+ + \lambda_{ab}(u^{ab})^\alpha + \lambda^a u^\alpha_a, \tag{D.4}$$

where $(\lambda^+, \lambda_{ab}, \lambda^a)$ transform in the $(\mathbf{1}, \bar{\mathbf{10}}, \mathbf{5})$ representation of U(5). It's a similar story for a spinor of opposite chirality, like w_α, except that the basis comes from applying 1, 3 and 5 γ^a's respectively to u^α_+. The spinor w_α splits into (w_+, w^{ab}, w_a), a $(\bar{\mathbf{1}}, \mathbf{10}, \bar{\mathbf{5}})$ representation of U(5). In calculating $\lambda^\alpha w_\alpha$ for example in terms of U(5) co-ordinates, we use the result that $u_+\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 u_+ = 1$ and that $u_+\gamma^a\gamma^b\gamma^c u_+$ and $u_+\gamma^a u_+$ vanish. Thus, the pure spinor constraints become

$$\lambda\gamma^a\lambda = \lambda^+\lambda^a + \frac{1}{8}\varepsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \tag{D.5}$$

$$\lambda\gamma_a\lambda = \lambda^b\lambda_{ab} = 0. \tag{D.6}$$

In the region defined by $\lambda^+ \neq 0$, we use equation (D.5) to write λ^a in terms of λ_+ and λ_{ab}

$$\lambda^a = \frac{-1}{8}(\lambda^+)^{-1}\varepsilon^{abcde}\lambda_{bc}\lambda_{de} = 0. \tag{D.7}$$

We then find that the expression for λ^a in equation (D.7) automatically satisfies the second condition (D.6), since, using the Young tableaux expression for tensors of specific symmetry properties, $\lambda_{a[b}\lambda_{cd}\lambda_{ef]} = k_1\lambda_{[ab}\lambda_{cd}\lambda_{ef]} + k_2\lambda_{(a[b)}\lambda_{cd}\lambda_{ef]} = 0$, where k_1 and k_2 are constants, and $\lambda_{(ab)} = 0$.

Since the ghosts λ^α are constrained, leaving 11 free complex parameters, i.e. $\lambda^\alpha = \lambda^\alpha(\lambda^+, \lambda_{ab})$, we expect some suitable constraints to be placed on w_α, such that it also has 11 complex degrees of freedom. Firstly, note that if we treat $\lambda\gamma^a\lambda = 0$ as first class constraints, recalling that $\lambda\gamma_a\lambda = 0$ are redundant for $\lambda^+ \neq 0$, the variation of w_a under a gauge transformation is given by

$$\delta w_a = -\varepsilon_a\lambda^+, \tag{D.8}$$

where $\varepsilon_a(\tau)$ is a local, bosonic parameter. Therefore, a good canonical gauge, which is both accessible and completely fixes the gauge symmetry, is given by

$$w_a = 0\,. \tag{D.9}$$

The constraints $\lambda\gamma^a\lambda = 0$ and $w_a = 0$ describe a second class constraint surface in the region $\lambda^+ \neq 0$. We parameterize the constraint surface using the co-ordinates λ^+, λ_{ab}, w_+, w_{ab}. The induced, Poisson bracket between the co-ordinates of the constraint surface needs to be calculated. The simplest way to calculate the bracket is to write the ghost action for the superparticle

$$S_g = \int d\tau \left(\dot{\lambda}^a w_a + \dot{\lambda}^+ w_+ + \frac{1}{2}\dot{\lambda}_{ab} w^{ab} - \Lambda_a \lambda\gamma^a\lambda - \Lambda^a \lambda\gamma_a\lambda \right), \tag{D.10}$$

then parameterize the second class constraint surface with $\lambda^+, \lambda_{ab}, w_+, w_{ab}$ so that

$$S_g = \int d\tau \left(\dot{\lambda}^+ w_+ + \frac{1}{2}\dot{\lambda}_{ab} w^{ab} \right). \tag{D.11}$$

It is clear to see that (λ^+, w_+) and (λ_{ab}, w^{ab}) are two conjugate pairs. The bracket between λ^α and w_β is given by

$$[\lambda^\alpha(\lambda^+, \lambda_{ab}), w_\beta(w_+, w^{ab})]_* = \delta^\alpha_\beta - u^\alpha_a v^a_\beta\,, \tag{D.12}$$

where $[\cdot,\cdot]_*$ is the induced Poisson bracket on the constraint surface, and where u^α_a and v^b_β are defined as basis spinors for λ^a and w_b respectively as in equation (D.4).

Instead of parameterizing the constraint surface, we could have alternatively defined a Dirac bracket.

E. $D = 10$, $N = 1$, Brink-Schwarz superparticle and super-Maxwell theory

E.1 $D = 10$, $N = 1$, Brink-Schwarz superparticle in the semi-light-cone gauge

For brevity, we directly specify the Brink-Schwarz superparticle in the semi-light-cone gauge, with no derivation. In this gauge, the fermionic, κ symmetry is gauge-fixed, but not the world-line reparameterization symmetry.

The system is defined by fundamental, bosonic operators $\hat{X}^m$, $\hat{P}_m$, and fermionic operators $\hat{S}^a$, $a = 1, \ldots, 8$, whose quantum commutator algebra is

$$\hat{X}^m \hat{P}_n - \hat{P}_n \hat{X}^m = i\delta^m_n\,, \qquad \hat{S}^a \hat{S}^b + \hat{S}^b \hat{S}^a = 2\delta_{ab}\,, \tag{E.1}$$

where all other brackets are zero. There is also the first class constraint $\hat{P}_m \hat{P}^m = 0$, which generates the world-line reparameterization symmetry and the hamiltonian is $\hat{P}_m \hat{P}^m/2$.

So $\hat{S}^a$ form a Clifford algebra and a representation can be built from the SO(8) Pauli matrices $\sigma^i_{a\dot{b}}$, described in appendix C. A generic wavefunction $\psi_{BS}(X)$ in the representation space is

$$\psi_{BS}(X) = e^{ik.X}(\varepsilon^i|i\rangle + \varepsilon^{\dot{a}}|\dot{a}\rangle)\,, \qquad k^m k_m = 0\,, \tag{E.2}$$

where ε^i, $\varepsilon^{\dot{b}}$ are bosonic constants, ε^i being a spin 1 SO(8) vector, and $\varepsilon^{\dot{a}}$ a spin 1/2 anti-chiral, SO(8) spinor, and where $i, \dot{a} = 1, \ldots, 8$. Also states $|i\rangle$ and $|\dot{a}\rangle$ are normalized as $\langle i|j\rangle = \delta_{ij}$, $\langle \dot{a}|\dot{b}\rangle = \delta_{\dot{a}\dot{b}}$, and $\hat{S}^a$ acts on ψ_{BS} as follows

$$\hat{S}^a \psi_{BS}(X) = e^{ik\cdot X}\left(\sigma^i_{a\dot{b}}\varepsilon^{\dot{b}}|i\rangle + \sigma^i_{a\dot{b}}\varepsilon^i|\dot{b}\rangle\right). \tag{E.3}$$

The inner product between two physical states ψ_{BS1} and ψ_{BS2} is given by

$$(\psi_{BS1}, \psi_{BS2}) \propto \left((\varepsilon_1^i)^*\varepsilon_2^i + (\varepsilon_1^{\dot{b}})^*\varepsilon_2^{\dot{b}}\right)\delta^{(10)}(k_1 - k_2). \tag{E.4}$$

We can show that $S^a = i2(2)^{\frac{1}{4}}(P^+)^{\frac{1}{2}}\theta^a$, where $\theta^\alpha = (\theta^a, \theta^{\dot{a}})$ is the usual fermionic, superspace variable and $P^\pm = (P^0 \pm P^9)/\sqrt{2}$.

The wavefunction ψ_{BS} corresponds, up to normalization constants, to the light-cone gauge, classical field multiplet of $D = 10$, $N = 1$ super Maxwell theory

$$\psi_{BS} = a^i|i\rangle - i2^{\frac{1}{4}}(P^+)^{-\frac{1}{2}}\chi^{\dot{b}}|\dot{b}\rangle, \tag{E.5}$$

where a^i and $\chi^{\dot{b}}$ behave like the light-cone gauge photon and photino fields of super-Maxwell. The normalization factor is included [25], so that the super-Maxwell and the BS superparticle supersymmetry transformation exactly coincides.

E.2 $D = 10$, $N = 1$ super-Maxwell

E.2.1 The action, symmetries and light-cone gauge fields

The SO(9,1) covariant action is the usual super Yang-Mills one, with U(1) gauge group

$$S = \int d^{10}X\left(\frac{i}{2}\chi\gamma^m\partial_m\chi - \frac{1}{4}f^2\right), \tag{E.6}$$

where χ^α is a Majorana-Weyl spinor, and where $f_{mn} = -ig[\partial_m a_n - \partial_n a_m]$ is the field strength for the U(1) gauge field a_m. The infinitesimal gauge symmetry is

$$\delta\chi^\alpha = 0 \quad \delta a_m = g\partial_m\phi(X), \tag{E.7}$$

and the action possesses the following supersymmetry,

$$\delta\chi^\alpha = -f_{mn}(\gamma^{mn}\varepsilon)^\alpha \quad \delta a_m = i(\varepsilon\gamma_m\chi), \tag{E.8}$$

for infinitesimal, fermionic, Majorana-Weyl constant ε.

To describe the physical modes, we choose the light-cone gauge $\partial_m a^m = 0$, $a^+ = 0$, where we assume again that $p^+ \neq 0$. In momentum space

$$a^- = \frac{1}{p^+}p^i a^i, \qquad \partial^2 a^i = 0, \tag{E.9}$$

thus the 8 massless, transverse modes a^i, describe the bosonic, physical sector. Also, χ^α obeys the Dirac equation $\gamma^m p_m\chi = 0$, which can be written as

$$\chi^a = -\frac{1}{(2)^{\frac{1}{2}}p^+}p^i\sigma^i_{a\dot{b}}\chi^{\dot{b}}, \qquad \partial^2\chi^{\dot{a}} = 0, \tag{E.10}$$

leaving the 8 massless modes $\chi^{\dot{a}}$, which describe the fermionic physical sector.

E.2.2 The superspace formulation

For details of $D = 10$, $N = 1$ superspace, super Yang-Mills see [26]. We only specify results relevant to this work.

The constraint equation is $F_{\alpha\beta} = 0$, where $F_{\alpha\beta}$ is the spin one superfield strength

$$F_{\alpha\beta} = D_\alpha A_\beta + D_\beta A_\alpha + 2i\gamma^m_{\alpha\beta} A_m \,, \tag{E.11}$$

and where D_α is the covariant superspace derivative of equation (2.6), and (A_α, A_m) the superspace, gauge connection. Since $F_{\alpha\beta}$ is a symmetric bispinor, it can be determined in terms of symmetric gamma matrices of the same chirality $F_{\alpha\beta} = F_m \gamma^m_{\alpha\beta} + F_{mnpqr}\gamma^{mnpqr}_{\alpha\beta}$, as in appendix C. Thus, the constraint equations split into two pieces, the first of which, $F_m = 0$, simply determines A_m in terms of A_α, and the second of which, $F_{mnpqr} = 0$, implies the equations of motion for A_α

$$\gamma^{\alpha\beta}_{mnpqr} D_\alpha A_\beta = 0 \,, \tag{E.12}$$

which have the effect of placing the theory on-shell. The equations of motion for the super field-strengths can be deduced from the Bianchi identities and the constraint equation [26]. They are

$$\gamma^m_{\alpha\beta}\partial_m W^\beta = 0 \,, \qquad \partial^m F_{mn} = 0 \,, \tag{E.13}$$

where W^α is the photino superfield-strength, given by $W^\alpha = (\gamma^m)^{\alpha\beta} F_{m\beta}/10$, and where $F_{mn} = \partial_m A_n - \partial_n A_m$ and $F_{\alpha m} = D_\alpha A_m - \partial_m A_\alpha$. The photino equation is equivalent to $\partial^m F_{m\alpha} = 0$.

In a θ^α expansion of the field strengths, the zero components are $F^{mn}|_{\theta=0} = f^{mn}$ and $W^\alpha|_{\theta=0} = \chi^\alpha$. We can choose a gauge, using $\delta A_\alpha = D_\alpha \phi$, such that

$$A_\alpha = -ia_m(X)\gamma^m_{\alpha\beta}\theta^\beta - \chi^\gamma(X)\gamma^m_{\alpha\beta}\gamma_{m\gamma\delta}\theta^\beta\theta^\delta + \cdots \,, \tag{E.14}$$

where a_m and χ^α obey the super-Maxwell equations of motion in the Lorentz gauge, $\partial^2 a_m = \partial^m a_m = 0$ and $\gamma^m_{\alpha\beta}\partial_m \chi^\beta = 0$. The $\cdots$ denote terms at higher order in θ which depend on space-time derivatives of a_m and χ^α.

References

[1] L. Brink and J.H. Schwarz, *Quantum superspace*, *Phys. Lett.* **B 100** (1981) 310.

[2] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, *J. High Energy Phys.* **09** (2001) 016 [hep-th/0105050].

[3] N. Berkovits, *Covariant quantization of the supermembrane*, *J. High Energy Phys.* **09** (2002) 051 [hep-th/0201151].

[4] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, *J. High Energy Phys.* **04** (2000) 018 [hep-th/0001035].

[5] N. Berkovits and B.C. Vallilo, *Consistency of super-Poincaré covariant superstring tree amplitudes*, *J. High Energy Phys.* **07** (2000) 015 [hep-th/0004171].

JHEP02(2004)011

[6] N. Berkovits, *Covariant quantization of the superstring*, *Int. J. Mod. Phys.* **A 16** (2001) 801 [hep-th/0008145].

[7] N. Berkovits and O. Chandia, *Lorentz invariance of the pure spinor BRST cohomology for the superstring*, *Phys. Lett.* **B 514** (2001) 394 [hep-th/0105149].

[8] N. Berkovits and P.S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, *Nucl. Phys.* **B 635** (2002) 75 [hep-th/0112160].

[9] N. Berkovits, *Relating the rns and pure spinor formalisms for the superstring*, *J. High Energy Phys.* **08** (2001) 026 [hep-th/0104247].

[10] N. Berkovits and O. Chandia, *Massive superstring vertex operator in $D = 10$ superspace*, *J. High Energy Phys.* **08** (2002) 040 [hep-th/0204121].

[11] M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, *The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings*, *Nucl. Phys.* **B 639** (2002) 182 [hep-th/0206104].

[12] P.A. Grassi, G. Policastro, M. Porrati and P. Van Nieuwenhuizen, *Covariant quantization of superstrings without pure spinor constraints*, *J. High Energy Phys.* **10** (2002) 054 [hep-th/0112162].

[13] P.A. Grassi, G. Policastro and P. van Nieuwenhuizen, *The massless spectrum of covariant superstrings*, *J. High Energy Phys.* **11** (2002) 001 [hep-th/0202123].

[14] P.A. Grassi, G. Policastro and P. van Nieuwenhuizen, *On the BRST cohomology of superstrings with/without pure spinors*, *Adv. Theor. Math. Phys.* **7** (2003) 499 [hep-th/0206216].

[15] P.S. Howe, *Pure spinors, function superspaces and supergravity theories in ten-dimensions and eleven-dimensions*, *Phys. Lett.* **B 273** (1991) 90.

[16] P.S. Howe, *Pure spinors lines in superspace and ten-dimensional supersymmetric theories*, *Phys. Lett.* **B 258** (1991) 141.

[17] I.A. Batalin, S.L. Lyakhovich and I.V. Tyutin, *Split involution and second class constraints*, *Mod. Phys. Lett.* **A 7** (1992) 1931.

[18] M. Henneaux and C. Teitelboim, *Quantization of gauge systems.* Princeton University Press, Princeton 1992.

[19] M. Chesterman, *Ghost constraints and the covariant quantization of the superparticle in ten dimensions.* Ph.D. thesis, Queen Mary, London, 2002.

[20] K. Harada and H. Mukaida, *Gauge invariance and systems with second class constraints*, *Z. Physik* **C 48** (1990) 151.

[21] P. Mitra and R. Rajaraman, *New results on systems with second class constraints*, *Ann. Phys. (NY)* **203** (1990) 137.

[22] A.S. Vytheeswaran, *Gauge invariances in second class constrained systems: a comparative look at two methods*, hep-th/9904084.

[23] I. Batalin, S. Lyakhovich and R. Marnelius, *Projection operator approach to general constrained systems*, *Phys. Lett.* **B 534** (2002) 201 [hep-th/0112175].

[24] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637 [hep-th/9207094].

JHEP02(2004)011

[25] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring theory. Vol. 1: Introduction*, Cambridge University Press, Cambridge 1987.

[26] E. Witten, *Twistor-like transform in ten-dimensions*, *Nucl. Phys.* **B 266** (1986) 245.